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FINITE SYMMETRIC SYSTEMS AND THEIR ANALYSIS

SOLOMON DINKEVICH
EBASCO Services, Inc., New York, N.Y. 10048, U.S.A.

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Abstract—Symmetry of linear mechanical systems permits one to substantially reduce the effort in their analysis. Classification and structural analysis of the finite linear symmetric systems (models) are studied in this paper. There are two variants of the symmetry approach: mechanical and algebraic. In accordance with the former, a symmetric system is replaced by one or several small nonsymmetric subsystems which are subjected to special loads obtained from the initial set. The total response of the original symmetric system is found by special superposition of partial responses of these subsystems. The algebraic approach is based on the explicit block diagonal decomposition of the matrix equation corresponding to a symmetric system. While both approaches have the same efficiency the latter is easier to implement and describe. It is presented here. The condtions under which the symmetry technique may be utilized do not include symmetry of the applied loads (specifically, symmetry of those loads which form the right side of the associated equations). Nevertheless, if the loads are symmetric, the efficiency of the symmetry approach substantially increases. Group theory, which is widely used in this paper, is the mathematical tool for the study of symmetry, and all necessary notions are introduced.

1. INTRODUCTION

1.1. The most time-consuming part of the analysis of large linear mechanical systems is associated with the construction and the solution of systems of linear equations. Even with the assistance of modern computers this problem is still of a great value. Fortunately, for many technological requirements large mechanical systems are often composed of a huge number of identical elements or subsystems. Many types of these "ordering" systems permit special, very efficient, methods of analysis. Symmetric systems form an important class of "ordering" systems.

Symmetry of the mechanical system is usually associated with its geometry. System S will be called symmetric (more precisely geometrically symmetric) if there exist rigid body motions known as symmetry transformations (or operations) which bring S into coincidence with itself with no breaks and/or intersections. There are three basic symmetry operations: rotations about axes, reflections in planes and translations. Two hundred and thirty different types of symmetric systems (SS) are left unaltered under these symmetry transformations. Every such system is infinitely large in the directions which possess the property of translational symmetry.

In this paper, the study of the SS is limited to a set of the finite symmetric systems which includes 14 different types. For such a purpose one has to eliminate translations from the set of symmetry operations under consideration.

1.2. The symmetry technique in the analysis of mechanical systems is utilized in two ways, which can be called the mechanical and the algebraic approaches. According to the former, the original symmetric system is replaced by one or several small, nonsymmetric subsystems. One has to build the special loads which these subsystems are subjected to and calculate the partial responses using the standard technique. The total response of the original system is then determined by a special superposition of the partial responses.

The algebraic approach, presented in this paper, is based on the explicit block diagonal decomposition of the matrix equation corresponding to the original symmetric system. While both approaches have the same efficiency, it seems that the latter is easier to describe and implement.

1.3. Group theory is the mathematical tool for the study of symmetry. It is widely used in quantum mechanics and crystallography; see Rosen (1983), Wigner (1959), Hamermesh

(1962), Falicov (1966), and Landau and Lifshitz (1977). Its first application to structural mechanics was made by Wigner in 1930. The next 50 years brought many publications in this field: Singh and Mishra (1972), Kardestuncer and Berg (1974), Miller (1981), Zhong and Qiu (1983), Burishkin and Gordeev (1984), Dinkevich (1984), and others. Note that in the case of simple finite symmetric systems the symmetry technique may be utilized with no involvement of group theory: see MacNeal *et al.* (1973) and Dinkevich (1977). It was even concluded that the finite element analysis of symmetric systems does not require the use of group theory (Everstine, 1987), which is certainly incorrect if we want to exploit symmetry systematically and completely. Group theory and especially group representation theory will be widely used in this paper. Although it is assumed that the readers are familiar with this subject, all necessary notions are introduced throughout the paper.

2. FINITE SYMMETRIC SYSTEMS

2.1. Finite symmetric systems (FSS) are those which are left unaltered under such symmetry operations as rotations about axes and reflections in planes. Symmetry operations will be denoted by g_1, g_2, etc., and g_1 will always present the identity transformation which leaves system S unmoved: $g_1 \equiv e$. A successive application of two symmetry operations g_i and g_j is also a symmetry operation, denoted by the product $g_i g_j$ if g_j precedes g_i or by $g_j g_i$ if g_j follows g_i. In general $g_i g_j \neq g_j g_i$, otherwise operations g_i and g_j are called commutative. Suppose that under rotation $g_j = c_n^j, j = 1, \ldots, n$ through the angle $\alpha_j = (j-1) \cdot 2\pi/n$ about an axis c, system S is left unaltered. Then, this axis is called the n-fold rotational axis and is denoted by c_n (n is its order). The product of two rotations, not necessarily about the same axis, is a new rotation: $c_n^i c_m^j = c_p^k$. The product of two reflections ($g \equiv \sigma$) in the same plane returns S to the initial state and is treated as the identity transformation: $\sigma \cdot \sigma \equiv \sigma^2 = e$. Rotations through angles $\alpha_1 = 0$ and $\alpha_{n+1} = 2\pi$ also return system S to its initial state, hence $c_n^1 = c_n^{n+1} = e$. The product of rotation c_n^j about an axis c_n and reflection σ in the perpendicular plane, that is σc_n^j, is also a symmetry transformation, known as a rotation reflection (or improper rotation) about a rotation reflection axis s_{2n}. Subscript $2n$ emphasizes the fact that such an axis is always of an even order. Rotation reflections are denoted by s_{2n}^j. Operations c_n^j and σ commute, hence $s_{2n}^j = \sigma c_n^j = c_n^j \sigma$.

Axes c_n and s_{2n} and planes σ are called the symmetry elements of systems S. Let A_1 be a certain point of S which does not belong to any symmetry element; then under h symmetry operations this point will be located at h different places $A_j = g_j A_1, j = 1, \ldots, h$. It is evident that under h symmetry transformations subsystem S_1 containing a neighbourhood of this point will occupy h different parts of S. Thus, if under h symmetry operations system S coincides with itself, it must be composed of h identical subsystems $S_j, j = 1, \ldots, h$. We call them the primitives, since they possess no symmetry elements. The S_1, chosen arbitrarily, is called the fundamental primitive. Thus

$$S = \bigcup_{j=1}^{h} S_j. \tag{1}$$

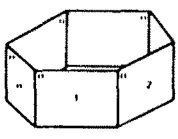
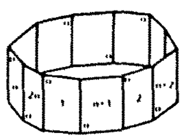
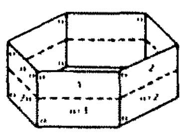
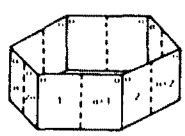
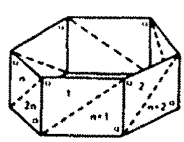
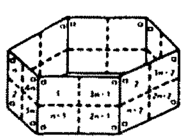
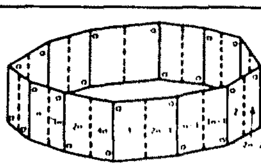
2.2. It is convenient to classify finite symmetric systems into three symmetry classes with respect to the order of their axes. The first (or lowest) class includes the FSS which have 2-fold axes and, possibly, some planes of reflection. The second (or middle) class is composed of systems which possess one n ($\geqslant 3$)-fold axis, a principal axis, and possibly, some second-order axes as well as symmetry planes. We call them the cyclically symmetric systems, the CSS. The third (or highest) class comprises systems which have several axes of order $n \geqslant 3$. We associate them with five Platonic solids, or regular polyhedra, namely, tetrahedrons (containing four regular triangular faces), hexahedrons or cubes (six square faces), octahedrons (eight regular triangular faces), dodecahedrons (12 regular pentagonal faces), and icosahedrons (20 regular triangular faces). In this paper we are interested in the second class of symmetric systems since it is mostly used in practice.

There are seven types of the CSS which we will denote as C_n, S_{2n}, C_{nh}, C_{nv}, D_n, D_{nh} and D_{nd}. They are distinguished in types and numbers of symmetry elements and are shown in Table 1 (column 4). The types and numbers of symmetry elements are given in column 3, where c_n and s_{2n} are the symmetry axes, and σ_h, σ_v and σ_d are the symmetry planes. σ_h represents a horizontal plane (in general, the plane which is perpendicular to the principal axis), σ_v indicates vertical planes (i.e. planes passing through the principal axis), and σ_d denotes the diagonal planes, that is, the vertical planes which do not pass through the horizontal axes c_2 but pass midway between each pair of them.

Having a principal axis (c_n or s_{2n}), every CSS is composed of n or $2n$ identical faces. Each face may or may not be symmetric; their symmetry is marked by small circles ("holes") in column 4. Possessing no symmetry elements, nonsymmetric faces will be called the primitives. Symmetric faces are divided into p primitives, whose total number is

$$h = pn, \quad p = 1, 2 \quad \text{or} \quad 4. \tag{2}$$

Table 1. Cyclically symmetric systems (CSS)

#	TYPE OF CSS	SYMMETRY ELEMENTS	CONFIGURATION	ORDER h	SYMMETRY OPERATIONS			
					g_μ $\mu=1,\ldots,n$	$g_{\mu+n}$ $\mu=1,\ldots,n$	$g_{\mu+2n}$ $\mu=1,\ldots,n$	$g_{\mu+3n}$ $\mu=1,\ldots,n$
1	2	3	4	5	6	7	8	9
1	C_n	$1\cdot c_n$		n	$c_n^{\mu-1}$			
2	S_{2n}	$1\cdot s_{2n}$		2n	$c_n^{\mu-1}$	$s_n^{\mu-1}$		
3	C_{nh}	$1\cdot c_n+1\cdot\sigma_h$		2n	$c_n^{\mu-1}$	$c_n^{\mu-1}\sigma_h$		
4	C_{nv}	$1\cdot c_n+n\cdot\sigma_v$		2n	$c_n^{\mu-1}$	$c_n^{\mu-1}\sigma_v$		
5	D_n	$1\cdot c_n+n\cdot c_2$		2n	$c_n^{\mu-1}$	$c_n^{\mu-1}u_2$		
6	D_{nh}	$1\cdot c_n+n\cdot c_2$ $+\ 1\cdot\sigma_h+n\cdot\sigma_v$		4n	$c_n^{\mu-1}$	$c_n^{\mu-1}\sigma_h$	$c_n^{\mu-1}u_2$	$c_n^{\mu-1}\sigma_v$
7	D_{nd}	$1\cdot c_n+n\cdot c_2$ $+\ 1\cdot\sigma_h+n\cdot\sigma_d$		4n	$c_n^{\mu-1}$	$s_n^{\mu-1}$	$c_n^{\mu-1}\sigma_d$	$c_n^{\mu-1}u_2$

As shown in Table 1 (column 5), $p = 1$ corresponds to system C_n, the simplest CSS. It has no other symmetry elements but the principal axis c_n. System S_{2n} also has nonsymmetric faces, which are arranged differently from those in C_n (as illustrated by holes). Their number is $2n$, hence $p = 2$. Each of n faces of the next three systems, C_{nh}, C_{nv} and D_n, possesses one additional symmetry element: a horizontal plane σ_h or a vertical plane σ_v, or a horizontal axis c_2. Therefore every face of these systems is divided into $p = 2$ primitives depicted in Table 1 by dotted lines. Faces of D_{nh} contain three symmetry elements each, namely, σ_h, σ_v and c_2, hence they are divided into $p = 4$ primitives. Finally, the last system D_{nd} can be derived from D_n by adjoining to it n vertical planes $\sigma_d^{(1)}, \ldots, \sigma_d^{(n)}$, which do not pass through horizontal axes $c_2^{(1)}, \ldots, c_2^{(n)}$ but midway between each two. In this case the principal axis c_n becomes a rotational reflection axis s_{2n}; therefore, it is easier to deduce system D_{nd} from S_{2n} by adding n vertical planes $\sigma_d^{(1)}, \ldots, \sigma_d^{(n)}$ passing through midlines of opposite faces of S_{2n} as is done in Table 1. Each of the $2n$ faces of D_{nd} is divided into two primitives whose total number is $4n$, hence $p = 4$.

2.3. The numbering sequence of faces and primitives of the CSS (see column 4 of Table 1) is chosen to satisfy the following cyclic rule: (a) n faces of systems C_n, C_{nh}, C_{nv}, D_n and D_{nh} are labeled by subscript μ from 1 to n counterclockwise. Systems S_{2n} and D_{nd} have $2n$ faces. We distinguish among their "top" and "bottom" faces. Top faces are marked by holes on the top and are numbered by $\mu = 1, \ldots, n$, while bottom faces are marked by holes on the bottom and are labeled by $\mu + n$ in the same direction; (b) primitives of C_n and S_{2n} coincide with their faces and are labeled accordingly. The first primitive of the μ-th face of systems C_{nh}, C_{nv}, D_n and D_{nh} has the same subscript μ, others — $\mu+n$ (systems C_{nh}, C_{nv} and D_n) or $\mu+n$, $\mu+2n$, and $\mu+3n$ (system D_{nh}). They are also enumerated in the counterclockwise direction. Each face of D_{nd} consists of two primitives, left and right. Left top primitives of D_{nd} are labeled by μ, left bottom — by $\mu+n$, right top — by $\mu+2n$, and right bottom primitives — by $\mu+3n$.

According to (1) any finite symmetric system is a union of h primitives, in the case of the CSS $h = pn$ and one can write

$$S = \bigcup_{\mu=1}^{n} \bigcup_{\nu=1}^{p=h/n} S_{\mu+(\nu-1)n}. \tag{3}$$

Symmetry operations are numbered according to the same cyclic rule, hence under operation g_j the fundamental primitive S_1 coincides with the primitive S_j

$$S_j = g_j S_1, \quad j = 1, \ldots, h \tag{4}$$

or

$$S_{\mu+(\nu-1)n} = g_{\mu+(\nu-1)n} S_1, \quad \mu = 1, \ldots, n; \quad \nu = 1, \ldots, p = h/n. \tag{5}$$

It follows from the cyclic rule that the first n symmetry operations of any CSS are rotations (more precisely, proper rotations) $c_n^{\mu-1}$ through the angles $(\mu-1)\cdot 2\pi/n$ about the principal axis. Other operations (see Table 1, columns 6–9) are more complicated; they are products of basic operations $c_n^{\mu-1}$ with additional operations as reflections in planes σ_h, σ_v or σ_d, or rotations about horizontal axes c_2 (these rotations through 180° are denoted by u_2): $c_n^{\mu-1}\sigma_h$, $c_n^{\mu-1}\sigma_v$, $c_n^{\mu-1}\sigma_d$, and $c_n^{\mu-1}u_2$†.

Both notations, (4) and (5), will be used in the further discussion: (4) as a rule in algebraic manipulations while (5) in most tables, since it contains more detail. Clearly

$$j \equiv \mu+(\nu-1)n, \quad j = 1, \ldots, h; \quad \mu = 1, \ldots, n; \quad \nu = 1, \ldots, p = h/n. \tag{6}$$

† It is necessary to emphasize that in all of these products a proper rotation $c_n^{\mu-1}$ follows reflections σs or a rotation u_2. Also note that while the operation σ_h commutes with any other operation, $\sigma_h g_j = g_j \sigma_h$, operations σ_v, σ_d, and u_2 $(= \sigma_h \sigma_v)$ do not commute with operations of other type. For example, $c_n^{\mu-1}\sigma_v = \sigma_v c_n^{-(\mu-1)} = \sigma_v c_n^{n-\mu+1}$. According to this, operations σ_v, σ_d, and u_2 may be called anti- or skew-commutative.

3. SYMMETRY GROUPS AND MATRICES G_w

3.1. Symmetry operations form a group $G = \{g_i\}_{i=1}^{h}$ of order h, i.e. a set of elements $g_1, g_2, \ldots, g_h$ satisfying the following four conditions (or group postulates): (1) closure: $g_i g_j$ belongs to G, (2) associativity: $g_i(g_j g_k) = (g_i g_j)g_k$, (3) existence of the identity: $g_1 = e$, and (4) existence of inverse elements $g_i^{-1} \in G$ such that $g_i g_i^{-1} = g_i^{-1} g_i = g_1$. If all elements of a group commute: $g_i g_j = g_j g_i$, $i, j = 1, \ldots, h$, then G is called commutative or Abelian, otherwise it is called non-Abelian. If a subset of elements $g_1, \ldots, g_h$ of group G also form a group, say, G_1, then G_1 is called a subgroup of G. One group may have several subgroups, and a given element of a group may appear in different subgroups. For example, $g_1 = e$ is the first element of each subgroup of group G.

Symmetry operations of the FSS comprise finite (if $h < \infty$) or continuous groups. These groups are known as point groups because any finite symmetric system has at least one fixed point which does not move under any symmetry transformation (such points belong to axes or planes, or to their intersections). Since a trivial group $G = \{e\}$ can be associated with an asymmetric system, we define a finite symmetric system as that which possesses a point group of order $h > 1$. Symbols C_n, $S_{2n}, \ldots, D_{nd}$ which we used to denote the CSS are introduced by Schoenflies for the corresponding point groups (Hamermesh, 1962; Falicov, 1966). Hence system C_n possesses point group C_n (known as a cyclic group), system S_{2n}—group S_{2n}, and so on. Continuous point groups C_∞, $C_{\infty h}$, $C_{\infty v}$, D_∞, and $D_{\infty h}$ are symmetry groups of the CSS with the principal axis of complete axial symmetry. It is convenient to treat them as the limit case of the finite point groups, $n \to \infty$. We will study the FSS with finite symmetry groups. Note that in accordance with Table 1 (columns 6–9) group C_n is the highest subgroup of all point groups, group S_{2n} is a subgroup of D_{nd}, groups C_{nh}, C_{nv} and D_n are subgroups of group D_{nh}, etc.

3.2. In the previous section symmetric systems are classified with respect to their symmetry elements (axes and planes). Here we continue their description based on the symmetry operations. Let S be a certain FSS: $S = \cup_{i=1}^{h} S_i$, (1). Introduce the identical FSS, $S' = \cup_{j=1}^{h} S'_j$ and superimpose them so that the fundamental primitive S'_1 of system S' is carried into S_i of S†. Then the primitive S'_j of S' will coincide with a certain primitive of S, say S_w or, more precisely, with $S_{w(j,i)}$ because the integer w $(1 \leqslant w \leqslant h)$ depends on j and i. In accordance with (4), $S'_j = S_{w(j,i)} = g_{w(j,i)}S_1$; on the other hand, $S'_j = g_j S'_1 = g_j S_i = g_j g_i S_1$. Hence

$$g_{w(j,i)} = g_j g_i, \quad i, j = 1, \ldots, h \tag{7}$$

with the property

$$w(j, 1) = j, \quad w(1, i) = i. \tag{8}$$

Symmetry of the FSS (i.e. the structure of the associated symmetry group) is completely described by products (7) which form a special $h \times h$ matrix known as the group (multiplication) table. We will use it in a transposed form and call the matrix G_w:

$$G_w = [g_{w(j,i)}]_{i,j=1}^{h} = [g_j g_i]_{i,j=1}^{h} = \left[\begin{array}{c|c|c|c|c} g_1 & \cdots & g_i & \cdots & g_h \\ \hline g_2 & \cdots & g_i g_2 & \cdots & g_h g_2 \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline g_h & \cdots & g_i g_h & \cdots & g_h^2 \end{array}\right]. \tag{9}$$

† Recalling that $S_i = g_i S_1$, we have to note that if operation g_i is a proper rotation c_n^j, then system S' is an exact copy of S. However, if g_i is a reflection σ, or a rotation u_2 or their products with c_n^j then system S' must be a mirror reflection copy of S; otherwise, its primitives will be labeled inconsistently with primitives of S. For example, if $S = D_{nh}$, one has to introduce four copies of D_{nh}: an exact copy S', two reflection copies S'' and S''' corresponding to σ_h and σ_v, respectively, and S^{IV} obtained from S by rotation about c_2. Since $u_2 = \sigma_n \sigma_v$ system S^{IV} is a double reflection copy of S.

3.3. One can write $(h-1)!$ matrices G_u for the same group $G = \{g_i\}_{i=1}^{h}$ since there are $(h-1)!$ sequences of h group elements ($g_1 \equiv e$ by convention). The cyclic rule for labeling of the primitives and symmetry operations described in Section 2 reduces this number to one. Moreover, it permits one to present matrices G_u (9) explicitly. To do so we introduce the parameter $r \equiv (v-1)n$ which is equal to 0, n, $2n$ and $3n$ ($v = 1, 2, 3, 4$) and six circulant matrices $B_r^{(k)}$, $k = 1, 2, \ldots, 6$, of order n. Each matrix contains only n different elements $h_{1+r}, h_{2+r}, \ldots, h_{n+r}$; we call them the basic elements:

$$B_r^{(1)} = \left[\begin{array}{c|c|c|c} h_{1+r} & h_{2+r} & \cdots & h_{n+r} \\ \hline h_{n+r} & h_{1+r} & \cdots & h_{n-1+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline h_{2+r} & h_{3+r} & \cdots & h_{1+r} \end{array}\right] \tag{10}$$

$$B_r^{(2)} = \left[\begin{array}{c|c|c|c} h_{1+r} & h_{n+r} & \cdots & h_{2+r} \\ \hline h_{2+r} & h_{1+r} & \cdots & h_{3+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline h_{n+r} & h_{n-1+r} & \cdots & h_{1+r} \end{array}\right] \tag{11}$$

$$B_r^{(3)} = \left[\begin{array}{c|c|c|c} h_{n+r} & h_{n-1+r} & \cdots & h_{1+r} \\ \hline h_{1+r} & h_{n+r} & \cdots & h_{2+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline h_{n-1+r} & h_{n-2+r} & \cdots & h_{n+r} \end{array}\right] \tag{12}$$

$$B_r^{(4)} = \left[\begin{array}{c|c|c|c} h_{n+r} & h_{1+r} & \cdots & h_{n-1+r} \\ \hline h_{n-1+r} & h_{n+r} & \cdots & h_{n-2+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline h_{1+r} & h_{2+r} & \cdots & h_{n+r} \end{array}\right] \tag{13}$$

$$B_r^{(5)} = \left[\begin{array}{c|c|c|c} h_{1+r} & h_{2+r} & \cdots & h_{n+r} \\ \hline h_{2+r} & h_{3+r} & \cdots & h_{1+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline h_{n+r} & h_{1+r} & \cdots & h_{n-1+r} \end{array}\right] \tag{14}$$

$$B_r^{(6)} = \left[\begin{array}{c|c|c|c|c} h_{2+r} & h_{3+r} & \cdots & h_{n+r} & h_{1+r} \\ \hline h_{3+r} & h_{4+r} & \cdots & h_{1+r} & h_{2+r} \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline h_{1+r} & h_{2+r} & \cdots & h_{n-1+r} & h_{n+r} \end{array}\right] \tag{15}$$

$$r = 0, n, 2n, 3n.$$

Matrices $B_r^{(5)}$ and $B_r^{(6)}$ are symmetric while $B_r^{(1)} - B_r^{(4)}$ are finite Toeplitz's matrices.

Substituting symmetry elements $g_{\mu+(v-1)n}$:

$$b_{\mu+r} = g_{\mu+(v-1)n}, \quad \mu = 1,\ldots,n; \quad v = 1,\ldots,p = h/n, \quad r = (v-1)n \tag{16}$$

we obtain six circulant matrices $G_r^{(k)}, k = 1,2,\ldots,6$ and use them to write down the explicit expression for matrices G_w (9):

$$G_w(C_n) = [G_0^{(5)}]_{h=n} \tag{17}$$

$$G_w(S_{2n}) = \left[\begin{array}{c|c} G_0^{(5)} & G_n^{(5)} \\ \hline G_n^{(5)} & G_0^{(6)} \end{array}\right]_{h=2n} \tag{18}$$

$$G_w(C_{nh}) = \left[\begin{array}{c|c} G_0^{(5)} & G_n^{(5)} \\ \hline G_n^{(5)} & G_0^{(5)} \end{array}\right]_{h=2n} = G_w(D_n) \tag{19}$$

$$G_w(C_{nv}) = \left[\begin{array}{c|c} G_0^{(5)} & G_n^{(1)} \\ \hline G_n^{(5)} & G_0^{(1)} \end{array}\right]_{h=2n} \tag{20}$$

$$G_w(D_{nh}) = \left[\begin{array}{c|c|c|c} G_0^{(5)} & G_n^{(5)} & G_{2n}^{(1)} & G_{3n}^{(1)} \\ \hline G_n^{(5)} & G_0^{(5)} & G_{3n}^{(1)} & G_{2n}^{(1)} \\ \hline G_{2n}^{(5)} & G_{3n}^{(5)} & G_0^{(1)} & G_n^{(1)} \\ \hline G_{3n}^{(5)} & G_{2n}^{(5)} & G_n^{(1)} & G_0^{(1)} \end{array}\right]_{h=4n} \tag{21}$$

$$G_w(D_{nd}) = \left[\begin{array}{c|c|c|c} G_0^{(5)} & G_n^{(1)} & G_{2n}^{(5)} & G_{3n}^{(1)} \\ \hline G_n^{(5)} & G_0^{(1)} & G_{3n}^{(5)} & G_{2n}^{(1)} \\ \hline G_{2n}^{(5)} & G_{3n}^{(4)} & G_0^{(6)} & G_n^{(1)} \\ \hline G_{3n}^{(5)} & G_{2n}^{(4)} & G_n^{(6)} & G_0^{(1)} \end{array}\right]_{h=4n}. \tag{22}$$

For $n = 4$ these matrices are presented in full in Table 2 where an integer j stands for a symmetry operation $g_j, j = 1,\ldots,h$. Matrices $G_w(C_n)$, $G_w(S_{2n})$ and $G_w(C_{nh})$ are symmetric which indicates that groups C_n, S_{2n} and C_{nh} are Abelian, groups C_{nv}, D_n and D_{nh} become Abelian only for $n = 2$ while D_{nd} is non-Abelian regardless of n.

Equations (17)–(22) may also be written in the form

$$G_w = \sum_{k=1}^{h} P(g_k)g_k, \tag{23}$$

where $P(g_k)$ are special permutation matrices of order h:

$$P(g_k) = [p_{ji}(g_k)]_{j,i=1}^{h} = [\delta_{k,w(j,i)}]_{j,i=1}^{h}. \tag{24}$$

Here $\delta_{k,w(j,i)}$ is the Kronecker delta and an integer $w(j,i)$ is defined by (7). Non-zero elements of matrix $P(g_k)$, which are equal to one, indicate all pairs (j,i), such that $g_j g_i = g_k$ where

Table 2. Matrices G_w, $n = 4$

$G_W(C_4)$

j	$g_j = c_4^{j-1}$			
	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

$G_W(S_8)$

j	$g_j = c_4^{j-1}$				$g_j = s_4^{j-5}$			
	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	6	7	8	2	3	4	1
6	6	7	8	5	3	4	1	2
7	7	8	5	6	4	1	2	3
8	8	5	6	7	1	2	3	4

$G_W(D_{4h})$

j	$g_j = c_4^{j-1}$				$g_j = c_4^{j-5}\sigma_h$				$g_j = c_4^{j-9}u_2$				$g_j = c_4^{j-13}\sigma_v$			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	3	4	1	6	7	8	5	12	9	10	11	16	13	14	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	1	2	3	8	5	6	7	10	11	12	9	14	15	16	13
5	5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	6	7	8	5	2	3	4	1	16	13	14	15	12	9	10	11
7	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	8	5	6	7	4	1	2	3	14	15	16	13	10	11	12	9
9	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
10	10	11	12	9	14	15	16	13	4	1	2	3	8	5	6	7
11	11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
12	12	9	10	11	16	13	14	15	2	3	4	1	6	7	8	5
13	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4
14	14	15	16	13	10	11	12	9	8	5	6	7	4	1	2	3
15	15	16	13	14	11	12	9	10	7	8	5	6	3	4	1	2
16	16	13	14	15	12	9	10	11	6	7	8	5	2	3	4	1

$G_W(C_{4h})$

j	$g_j = c_4^{j-1}$				$g_j = c_4^{j-5}\sigma_h$			
	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	6	7	8	1	2	3	4
6	6	7	8	5	2	3	4	1
7	7	8	5	6	3	4	1	2
8	8	5	6	7	4	1	2	3

$G_W(D_{4d})$

j	$g_j = c_4^{j-1}$				$g_j = s_4^{j-5}$				$g_j = c_4^{j-9}\sigma_d$				$g_j = c_4^{j-13}u_2$			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	3	4	1	8	5	6	7	10	11	12	9	16	13	14	15
3	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
4	4	1	2	3	6	7	8	5	12	9	10	11	14	15	16	13
5	5	6	7	8	1	2	3	4	13	14	15	16	9	10	11	12
6	6	7	8	5	4	1	2	3	14	15	16	13	12	9	10	11
7	7	8	5	6	3	4	1	2	15	16	13	14	11	12	9	10
8	8	5	6	7	2	3	4	1	16	13	14	15	10	11	12	9
9	9	10	11	12	16	13	14	15	2	3	4	1	5	6	7	8
10	10	11	12	9	15	16	13	14	3	4	1	2	8	5	6	7
11	11	12	9	10	14	15	16	13	4	1	2	3	7	8	5	6
12	12	9	10	11	13	14	15	16	1	2	3	4	6	7	8	5
13	13	14	15	16	12	9	10	11	6	7	8	5	1	2	3	4
14	14	15	16	13	11	12	9	10	7	8	5	6	4	1	2	3
15	15	16	13	14	10	11	12	9	8	5	6	7	3	4	1	2
16	16	13	14	15	9	10	11	12	5	6	7	8	2	3	4	1

$G_W(C_{4v}) = G_W(D_4)$

j	$g_j = c_4^{j-1}$				$g_j = c_4^{j-5}\sigma_v$			
	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	8	5	6	7
3	3	4	1	2	7	8	5	6
4	4	1	2	3	6	7	8	5
5	5	6	7	8	1	2	3	4
6	6	7	8	5	4	1	2	3
7	7	8	5	6	3	4	1	2
8	8	5	6	7	2	3	4	1

NOTE: IN ALL GROUP TABLES G_W INTEGER j STATES FOR g_j.

Table 3. Matrices $P(g_{\mu+(\nu-1)n})$

#	TYPE OF CSS	MATRIX ORDER h	$P(g_\mu)$ $\mu=1,\dots,n$	$P(g_{\mu+n})$ $\mu=1,\dots,n$	$P(g_{\mu+2n})$ $\mu=1,\dots,n$	$P(g_{\mu+3n})$ $\mu=1,\dots,n$
1	C_n	n	$\hat{J}_\mu$		NOTE: $J_\mu = \begin{bmatrix} & I_{n+1-\mu} \\ I_{\mu-1} & \end{bmatrix}_n$, $I_\mu = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_\mu$	
2	S_{2n}	2n	$\begin{bmatrix} \hat{J}_\mu & \\ & \hat{J}_{\mu-1} \end{bmatrix}$	$\begin{bmatrix} & \hat{J}_\mu \\ \hat{J}_\mu & \end{bmatrix}$	$\hat{J}_\mu = \begin{bmatrix} \hat{I}_\mu & \\ & \hat{I}_{n-\mu} \end{bmatrix}_n$, $\hat{I}_\mu = \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & 1 & & \\ 1 & & & \end{bmatrix}_\mu$	
3	C_{nh}	2n	$\begin{bmatrix} \hat{J}_\mu & \\ & \hat{J}_\mu \end{bmatrix}$	$\begin{bmatrix} & \hat{J}_\mu \\ \hat{J}_\mu & \end{bmatrix}$	$J_1 = \hat{J}_n = I_n$	
4-5	C_{nv} AND D_n	2n	$\begin{bmatrix} \hat{J}_\mu & \\ & J_\mu \end{bmatrix}$	$\begin{bmatrix} & J_\mu \\ \hat{J}_\mu & \end{bmatrix}$		
6	D_{nh}	4n	$\begin{bmatrix} \hat{J}_\mu & & & \\ & \hat{J}_\mu & & \\ & & J_\mu & \\ & & & J_\mu \end{bmatrix}$	$\begin{bmatrix} & \hat{J}_\mu & & \\ \hat{J}_\mu & & & \\ & & & J_\mu \\ & & J_\mu & \end{bmatrix}$	$\begin{bmatrix} & & J_\mu & \\ & & & J_\mu \\ \hat{J}_\mu & & & \\ & \hat{J}_\mu & & \end{bmatrix}$	$\begin{bmatrix} & & & J_\mu \\ & & J_\mu & \\ & \hat{J}_\mu & & \\ \hat{J}_\mu & & & \end{bmatrix}$
7	D_{nd}	4n	$\begin{bmatrix} \hat{J}_\mu & & & \\ & J_\mu & & \\ & & \hat{J}_{\mu-1} & \\ & & & J_\mu \end{bmatrix}$	$\begin{bmatrix} & J_\mu & & \\ \hat{J}_\mu & & & \\ & & & J_\mu \\ & & \hat{J}_{\mu-1} & \end{bmatrix}$	$\begin{bmatrix} & & \hat{J}_\mu & \\ & & & J_\mu \\ J_\mu & & & \\ & J_{\mu+1} & & \end{bmatrix}$	$\begin{bmatrix} & & & J_\mu \\ & & J_\mu & \\ & J_{\mu+1} & & \\ \hat{J}_\mu & & & \end{bmatrix}$

k is fixed. Rewrite (23) in the following form

$$G_u = \sum_{\mu=1}^{n} \sum_{\nu=1}^{h/n} P(g_{\mu+(\nu-1)n}) g_{\mu+(\nu-1)n}. \tag{25}$$

Permutation matrices $P(g_{\mu+(\nu-1)n})$ are depicted in Table 3 for all seven cyclic groups.

4. SYMMETRY MATRICES

4.1. Introduce another set of h permutation $h \times h$ matrices

$$Q(g_j) = [q_{ik}(g_j)]_{i,k=1}^{h} = [\delta_{k,w(j,i)}]_{i,k=1}^{h}. \tag{26}$$

Non-zero elements of $Q(g_j)$ describe a sequence of the primitives $S_k \equiv S_{w(j,i)}$ of a finite symmetric system S which will coincide with the primitive S'_j (j is fixed) of the identical system S' when its fundamental primitive S'_1 successively coincides with all $S_i (i = 1, \dots, h)$ of S. Thus matrix $Q(g_j)$ can be treated as a special form of presentation of the j-th row of matrix G_w (9). Since the whole set of matrices $Q(g_j)$, $j = 1, \dots, h$ is isomorphic (i.e. in a one-to-one correspondence) with the matrix G_w of the FSS, it is proper to call $Q(g_j)$ the symmetry matrices.

4.2. Symmetry matrices are not necessarily symmetric. In fact, $q_{ik}(g_j) = \delta_{k,w(j,i)} \neq 0$ if $g_k = g_j g_i$, while $q_{ki}(g_j) = \delta_{i,w(j,k)} \neq 0$ if $g_i = g_j g_k$. Hence the reciprocal relation $q_{ik}(g_j) = q_{ki}(g_j)$ simply means that $g_k = g_j g_i = g_j^2 g_k$; that is, that $g_j^2 = e$. Thus matrix $Q(g_j)$ is symmetric if and only if $g_j^2 = e$, $j = 1, \dots, h$.

Notice the following properties of symmetry matrices $Q(g_j)$:

$$q_{1j}(g_j) = 1, \quad j = 1, \dots, h$$

$$Q(g_1) = I_n,$$

$$\sum_{j=1}^{h} q_{ik}(g_j) = 1, \quad i, k = 1, \dots, h. \tag{27}$$

Symmetry matrices corresponding to the CSS are shown in Table 4.

4.3. Table 4 is very similar to Table 3. However, symmetry matrices $Q(g_j)$ are much more interesting and important than matrices $P(g_j)$ of Table 3 because they possess the fundamental properties. To study them we have to introduce the notion of the group representations. It is said that the set A of h nonsingular $n \times n$ matrices A_j, $j = 1, \ldots, h$, form a (matrix) group under matrix multiplication, if these matrices satisfy the following group postulates: (1) closure: product $A_i A_j (i, j = 1, \ldots, h)$ belongs to set A, (2) associativity: $A_i(A_j A_k) = (A_i A_j)A_k (i, j, k = 1, \ldots, h)$, (3) existence of the identity: $A_1 = I_n$, and (4) existence of inverse matrices: A_i^{-1} belonging to A. Such a matrix group is called an n-dimensional representation of (abstract) group $G = \{g_j\}_{j=1}^h$ if matrices A_j are in a one-to-one correspondence with elements g_j, i.e. if $A_j \equiv A(g_j)$ and

$$\begin{gathered} A(g_i)A(g_j) = A(g_i g_j) \\ A(g_1) = I_n, \quad A(g_i^{-1}) = A^{-1}(g_i) \\ i, j = 1, \ldots, h. \end{gathered} \tag{28}$$

Furthermore, if there exists a matrix U such that

$$U^{-1}A(g_j)U = T(g_j), \quad j = 1, \ldots, h, \tag{29}$$

where $T(g_j)$ are block diagonal matrices of the same configuration, then the matrix representation A is called reducible, otherwise it is called irreducible. Denote by τ_r, $r = 1, \ldots, H$, the r-th irreducible representation of group $G = \{g_j\}_{j=1}^h$. It contains h unitary matrices (we consider finite groups) of order n_r:

$$\tau_r(g_j) = [\tau_{r\alpha\beta}(g_j)]_{\alpha,\beta=1}^{n_r}, \quad r = 1, \ldots, H; \quad j = 1, \ldots, h. \tag{30}$$

Elements $\tau_{r\alpha\beta}(g_j)$ are the roots of unity. If $\dim \tau_r (\equiv n_r) = 1$, $\tau_r(g_j) = \tau_{r11}(g_j)$, $j = 1, \ldots, h$. In such a case the second and third subscripts may be omitted. Let us fix the index j in eqn (30) and calculate the number of elements $\tau_{r\alpha\beta}(g_j)$ of all irreducible representations. Since $\alpha, \beta = 1, \ldots, n_r$ and $r = 1, \ldots, H$, it is equal to $\Sigma_{r=1}^H n_r^2$. In accordance with the Bernside's

Table 4. Symmetry matrices $Q(g_{\mu,\ldots,\mu+3n})$

#	TYPE OF CSS	MATRIX ORDER h	$Q(g_\mu)$ $\mu=1,\ldots,n$	$Q(g_{\mu+n})$ $\mu=1,\ldots,n$	$Q(g_{\mu+2n})$ $\mu=1,\ldots,n$	$Q(g_{\mu+3n})$ $\mu=1,\ldots,n$
1	C_n	n	J_μ		NOTE: $J_\mu = \begin{bmatrix} & I_{n+1-\mu} \\ I_{\mu-1} & \end{bmatrix}_n$, $I_\mu = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_\mu$; $J_1 = I_n$	
2	S_{2n}	2n	$\begin{bmatrix} J_\mu & \\ & J_\mu \end{bmatrix}$	$\begin{bmatrix} & J_\mu \\ J_{\mu+1} & \end{bmatrix}$		
3	C_{nh}	2n	$\begin{bmatrix} J_\mu & \\ & J_\mu \end{bmatrix}$	$\begin{bmatrix} & J_\mu \\ J_\mu & \end{bmatrix}$		
4-5	C_{nv} AND D_n	2n	$\begin{bmatrix} J_\mu & \\ & J_{n+2-\mu} \end{bmatrix}$	$\begin{bmatrix} & J_\mu \\ J_{n+2-\mu} & \end{bmatrix}$		
6	D_{nh}	4n	$\begin{bmatrix} J_\mu & & & \\ & J_\mu & & \\ & & J_{n+2-\mu} & \\ & & & J_{n+2-\mu} \end{bmatrix}$	$\begin{bmatrix} & J_\mu & & \\ J_\mu & & & \\ & & & J_{n+2-\mu} \\ & & J_{n+2-\mu} & \end{bmatrix}$	$\begin{bmatrix} & & J_\mu & \\ & & & J_\mu \\ J_{n+2-\mu} & & & \\ & J_{n+2-\mu} & & \end{bmatrix}$	$\begin{bmatrix} & & & J_\mu \\ & & J_\mu & \\ & J_{n+2-\mu} & & \\ J_{n+2-\mu} & & & \end{bmatrix}$
7	D_{nd}	4n	$\begin{bmatrix} J_\mu & & & \\ & J_{n+2-\mu} & & \\ & & J_\mu & \\ & & & J_{n+2-\mu} \end{bmatrix}$	$\begin{bmatrix} & J_\mu & & \\ J_{n+2-\mu} & & & \\ & & & J_\mu \\ & & J_{n+2-\mu} & \end{bmatrix}$	$\begin{bmatrix} & & J_\mu & \\ & & & J_{n+1-\mu} \\ J_{\mu+1} & & & \\ & J_{n+2-\mu} & & \end{bmatrix}$	$\begin{bmatrix} & & & J_\mu \\ & & J_{n+1-\mu} & \\ & J_{\mu+1} & & \\ J_{n+2-\mu} & & & \end{bmatrix}$

theorem (Falicov, 1966)

$$\sum_{r=1}^{H} n_r^2 = h. \tag{31}$$

Hence $H \leqslant h$. In fact $H = h$, if and only if, all irreducible representations are one-dimensional: $n_r = 1$, $r = 1, \ldots, H$. Since j runs from 1 to h, it follows from (31) that the total number of elements of all irreducible representations is equal to h^2. They may be grouped so as to form a special $h \times h$ matrix U in which the elements of each one-dimensional representation establish one column while elements of a n_r-dimensional representation form a set of n_r^2 consecutive columns. Each column of U is normalized to the unit length and $(n_r/h)^{1/2}$ is the normalization factor. Thus if τ_r is one-dimensional, the corresponding column of U is

$$u_r^{\mathrm{T}} = [(1/\sqrt{h})\tau_{r11}(g_1), \ldots, (1/\sqrt{h})\tau_{r11}(g_h)]; \tag{32}$$

if τ_r is n_r-dimensional, it forms columns $u_{r11}, u_{r12}, \ldots, u_{r21}, u_{r22}, \ldots, u_{rn_rn_r}$ and column $u_{r\gamma\alpha}$ is

$$u_{r\gamma\alpha}^{\mathrm{T}} = [(n_r/h)^{1/2}\tau_{r\gamma\alpha}(g_1), \ldots, (n_r/h)^{1/2}\tau_{r\gamma\alpha}(g_h)] \quad \alpha, \gamma = 1, \ldots, n_r. \tag{33}$$

It should then be evident that the i-th row of matrix U consists of the elements of all irreducible representations corresponding to the symmetry element g_i:

$$u_i = [(1/\sqrt{h})\tau_{111}(g_i), \ldots, (n_H/h)^{1/2}\tau_{Hn_Hn_H}(g_i)] \quad i = 1, \ldots, h. \tag{34}$$

Therefore, matrix U may be written in short as

$$U = [u_{is}]_{i,s=1}^{h} = \left[\left(\frac{n_r}{h}\right)^{1/2}[[\tau_{r\gamma\alpha}(g_i)]_{\alpha=1}^{n_r}]_{\gamma=1}^{n_r}\right]_{i,r=1}^{h,H}. \tag{35}$$

In this notation subscript s is associated with three subscripts r, γ, α: $s \Leftrightarrow (r\gamma\alpha)$. When α and γ run from 1 to n_r and r from 1 to H, subscript s runs from 1 to h in accordance with (31). Matrix U corresponding to group D_{3h} is given in full in Section 6.

Matrix elements $\tau_{r\alpha\beta}(g_i)$ of all irreducible representations of any group satisfy some orthogonality relations (Hamermesh, 1962; Falicov, 1966), in particular,

$$\sum_{r=1}^{H} \frac{n_r}{h} \sum_{\alpha,\beta=1}^{n_r} \tau_{r\alpha\beta}(g_i)\bar{\tau}_{r\alpha\beta}(g_k) = \delta_{ik}, \quad i, k = 1, \ldots, h \tag{36}$$

where $\bar{\tau}_{r\alpha\beta}(g_k)$ is a complex conjugate of $\tau_{r\alpha\beta}(g_k)$. Hence matrix U is uninormal

$$U^{-1} = U^{\mathrm{H}} \equiv \bar{U}^{\mathrm{T}}$$

(symbol "H" means Hermitian transpose) and

$$U^{\mathrm{H}} = [\bar{u}_{sk}]_{s,k=1}^{h} = \left[\left(\frac{n_r}{h}\right)^{1/2}[[\bar{\tau}_{r\gamma\beta}(g_k)]_{\beta=1}^{n_r}]_{\gamma=1}^{n_r}\right]_{r,k=1}^{H,h}. \tag{37}$$

The irreducible representations of point groups are known, their dimensions satisfy the inequality: $1 \leqslant n_r \leqslant 5$. The irreducible representations of groups corresponding to the CSS are one- and two-dimensional only, and are given in Table 5. By observing this table, one may conclude that Abelian (or commutative) groups C_n, S_{2n} and C_{nh} have only one-dimensional irreducible representations, while representations of non-Abelian groups C_{nv}, D_n, D_{nh} and D_{nd} are both one- and two-dimensional.

Table 5. Irreducible representations of groups C_n, S_{2n}, C_{nh}, C_{nv}, D_n, D_{nh} and D_{nd}

#	GROUP	ORDER h	EVENNESS OF n	ONE DIMENSIONAL IRR REPRESENTATIONS Γ_r	r	ELEMENTS OF Γ_r: $\Gamma_r(g_\mu)$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+n})$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+2n})$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+3n})$, $\mu=1,\ldots,n$	TWO DIMENSIONAL IRR REPRESENTATIONS Γ_r	r	MATRICES OF Γ_r: $\Gamma_r(g_\mu)$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+n})$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+2n})$, $\mu=1,\ldots,n$	$\Gamma_r(g_{\mu+3n})$, $\mu=1,\ldots,n$
1	C_n	n	ANY	Γ_r	$1,\ldots,n$	$\varepsilon^r_{\mu-1}$		NOTE: $\varepsilon^r_\mu = \exp(ir\mu 2\pi/n)$, $i=\sqrt{-1}$							
2	S_{2n}	$2n$	ANY	Γ_r	$1,\ldots,2n$	$\varepsilon^r_{\mu-1}$	$\varepsilon^r_{\mu-1/2}$	$E^r_{\mu-1} = \begin{bmatrix} \varepsilon^r_{\mu-1} & \\ & \bar{\varepsilon}^r_{\mu-1} \end{bmatrix}$, $\bar{E}^r_{\mu-1} = \begin{bmatrix} & \varepsilon^r_{\mu-1} \\ \bar{\varepsilon}^r_{\mu-1} & \end{bmatrix}$							
3	C_{nh}	$2n$	ANY	Γ_r	$1,\ldots,n$	$\varepsilon^r_{\mu-1}$	$\varepsilon^r_{\mu-1}$								
				Γ_{r+n}	$1,\ldots,n$	$\varepsilon^r_{\mu-1}$	$-\varepsilon^r_{\mu-1}$								
4-5	C_{nv} AND D_n	$2n$	EVEN	Γ_1		1	1			Γ_{r+4}	$1,\ldots,\frac{n}{2}-1$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$		
				Γ_2		1	−1								
				Γ_3		$(-1)^{\mu-1}$	$(-1)^{\mu-1}$								
				Γ_4		$(-1)^{\mu-1}$	$(-1)^{\mu}$								
			ODD	Γ_1		1	1			Γ_{r+2}	$1,\ldots,\frac{n-1}{2}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$		
				Γ_2		1	−1								
6	D_{nh}	$4n$	EVEN	Γ_1		1	1	1	1	Γ_{r+8}	$1,\ldots,\frac{n}{2}-1$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$
				Γ_2		1	1	−1	−1						
				Γ_3		1	−1	1	−1						
				Γ_4		1	−1	−1	1						
				Γ_5		$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$\Gamma_{r+\frac{n}{2}+7}$	$1,\ldots,\frac{n}{2}-1$	$E^r_{\mu-1}$	$-\bar{E}^r_{\mu-1}$	$-E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$
				Γ_6		$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu}$						
				Γ_7		$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu-1}$	$(-1)^{\mu}$						
				Γ_8		$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu}$	$(-1)^{\mu-1}$						
			ODD	Γ_1		1	1	1	1	Γ_{r+4}	$1,\ldots,\frac{n-1}{2}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$
				Γ_2		1	1	−1	−1						
				Γ_3		1	−1	1	−1	$\Gamma_{r+\frac{n+7}{2}}$	$1,\ldots,\frac{n-1}{2}$	$E^r_{\mu-1}$	$-\bar{E}^r_{\mu-1}$	$-E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$
				Γ_4		1	1	−1	1						
7	D_{nd}	$4n$	EVEN	Γ_1		1	1	1	1	Γ_{r+4}	$1,\ldots,n-1$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$	$E^r_{\mu-1/2}$	$\bar{E}^r_{\mu-1/2}$
				Γ_2		1	1	−1	−1						
				Γ_3		1	1	1	−1						
				Γ_4		1	−1	−1	1						
			ODD	Γ_1		1	1	1	1	Γ_{r+4}	$1,\ldots,\frac{n-1}{2}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$	$E^r_{\mu-2}$	$\bar{E}^r_{\mu-2}$
				Γ_2		1	1	−1	−1						
				Γ_3		1	−1	1	−1	$\Gamma_{r+\frac{n+7}{2}}$	$1,\ldots,\frac{n-1}{2}$	$E^r_{\mu-1}$	$\bar{E}^r_{\mu-1}$	$E^r_{\mu-2}$	$-\bar{E}^r_{\mu-2}$
				Γ_4		1	−1	−1	1						

4.4. The following two lemmas present the fundamental properties of the symmetry matrices $Q(g_j)$.

Lemma 1. Matrices $Q(g_j)$, $j = 1,\ldots,h$, comprise an h-dimensional representation Q of group $G = \{g_j\}_{j=1}^h$.

Proof. First, according to (27), $Q(g_1) = I_h$. Second, consider a matrix product $Q(g_s)Q(g_t)$ for g_s and g_t belonging to G:

$$Q(g_s)Q(g_t) = \left[\sum_{l=1}^{h} q_{il}(g_s)q_{lk}(g_t)\right]_{i,k=1}^{h}$$

$$= \left[\sum_{l=1}^{h} \delta_{l,w(s,i)}\delta_{k,w(t,l)}\right]_{i,k=1}^{h} = [\delta_{k,w(t,w(s,i))}]_{i,k=1}^{h}$$

because

$$\delta_{l,w(s,i)} = \begin{cases} 1 & \text{if } l = w(s,i) \\ 0 & \text{otherwise.} \end{cases}$$

It follows from associativity of the triple product

$$\begin{aligned} g_i g_j g_k &= (g_i g_j) g_k = g_{w(i,j)} g_k = g_{w(w(i,j),k)} \\ &= g_i (g_j g_k) = g_i g_{w(j,k)} = g_{w(i,w(j,k))} \end{aligned}$$

that

$$w(i, w(j, k)) = w(w(i, j), k) \quad i, j, k = 1, \dots, h$$

and therefore

$$\begin{aligned} Q(g_s)Q(g_t) &= [\delta_{k,w(w(t,s),i)}]^h_{i,k=1} = [q_{ik}(g_{w(t,s)})]^h_{i,k=1} \\ &= Q(g_{w(t,s)}) = Q(g_t g_s), \quad s, t = 1, \dots, h, \end{aligned} \tag{38}$$

which means that $Q(g_s)Q(g_t)$ belongs to the set Q. This equation does not contradict (28) because matrices of the group representations are conventionally written in a transposed form so that, for example, a matrix–vector product is written as $x^T A$. Therefore, if we denote $Q(g_s) = A^T(g_s)$, $s = 1, \dots, h$, then (38) will take the standard form (28). Next, let g_t be equal to g_s^{-1}. Then $Q(g_s)Q(g_s^{-1}) = Q(g_1) = I_h$ and

$$Q^{-1}(g_s) = Q(g_s^{-1}), \quad s = 1, \dots, h. \tag{39}$$

Finally, $[Q(g_s)Q(g_t)]Q(g_u) = Q(g_s)[Q(g_t)Q(g_u)]$ by the associativity property of a matrix product.

□

In fact, symmetry matrices $Q(g_j)$, $j = 1, \dots, h$ form a special reducible representation of group $G = \{g_j\}^h_{j=1}$ known as the regular representation (Falicov, 1966). This follows from

Lemma 2. There exists the explicit block diagonal decomposition of matrices $Q(g_j)$, $j = 1, \dots, h$:

$$Q(g_j) = UT(g_j)U^H, \quad j = 1, \dots, h, \tag{40}$$

where U and U^H are defined by (35) and (37), respectively, and $T(g_j)$ is a block diagonal unitary matrix:

$$T(g_j) = \begin{bmatrix} \underbrace{\tau_1(g_j) \ \ddots \ \tau_1(g_j)}_{n_1 \text{ times}} & & & \\ & \underbrace{\tau_2(g_j) \ \ddots \ \tau_2(g_j)}_{n_2 \text{ times}} & & \\ & & \ddots & \\ & & & \underbrace{\tau_H(g_j) \ \ddots \ \tau_H(g_j)}_{n_H \text{ times}} \end{bmatrix} \tag{41}$$

$$j = 1, \dots, h$$

or in short

$$T(g_j) = [[\tau_r(g_j)\delta_{\alpha\beta}\delta_{rt}]^{n_r}_{\alpha,\beta=1}]^{H}_{r,t=1}, \quad j = 1,\ldots,h. \tag{42}$$

Proof. Equation (40) is the matrix form of the following orthogonality relation

$$\sum_{r=1}^{H} \frac{n_r}{h} \sum_{\alpha,\beta,\gamma=1}^{n_r} \tau_{r\gamma\alpha}(g_i)\tau_{r\alpha\beta}(g_j)\bar{\tau}_{r\gamma\beta}(g_k) = \delta_{k,w(i,j)}, \quad i,j,k = 1,\ldots,h \tag{43}$$

which generalizes the identity (36).

□

Thus besides the uninormal matrix U, h^2 matrix elements $\tau_{r\alpha\beta}(g_j)$ of all irreducible representations of group G form h block diagonal unitary matrices $T(g_j)$ and all of them satisfy eqn (40).

5. MATRICES INTRODUCED ON THE FSS

5.1. In constructing matrices corresponding to the FSS, we must obey the special symmetry law [derived from eqn (4): $S_j = g_jS_1$, $j = 1,\ldots,h$] which states that we are free to introduce the mesh and all variables and functions on the fundamental primitive S_1 only. The mesh, variables and functions associated with the primitive S_j must be obtained by applying the symmetry operation g_j to the analogous characteristics of S_1, $j = 1,\ldots,h$. It is convenient to present all matrices introduced on the FSS in a block (partitioned) form associating the blocks (submatrices) with the primitives. Let A_* be such a block matrix. Then it is of order mh and its blocks A_{ik}, $i, k = 1,\ldots,h$ are of order m, where h is the number of primitives and m is the total number of variables (for instance, degrees of freedom) of any single primitive:

$$A_* = [A_{ik}]^h_{i,k=1} = [[a_{ik\sigma\tau}]^m_{\sigma,\tau=1}]^h_{i,k=1}. \tag{44}$$

Theorem 1. *Matrix* A_* (44) *introduced on the FSS has the following presentation:*

$$A_* = \sum_{j=1}^{h} Q(g_j) \otimes A_{1j}, \tag{45}$$

where $Q(g_j)$ *are symmetry matrices* (26), A_{1j} *are blocks of the first block row of* A_* *and the symbol* $\otimes$ *stands for Kronecker multiplication*†.

Proof. Once again we consider two identical FSS, namely, $S = U^h_{j=1}S_j$ and $S' = U^h_{j=1}S'_j$ and coincide them so that $S'_1 = S_i$ and $S'_j = S_{w(j,i)}$. Suppose that matrix $A_* = [A_{pj}]^h_{p,j=1}$ corresponds to system S while matrix $A'_* = [A'_{pj}]^h_{p,j=1}$ to S'. Then

$$A'_{pj} = A_{w(p,i),w(j,i)}, \quad p, j = 1,\ldots,h, \quad i \text{ is fixed}.$$

On the other hand, $A'_* = A_*$ because both systems are identical, hence

$$A'_{pj} = A_{pj}, \quad p, j = 1,\ldots,h.$$

† If B and C are two arbitrary $p \times q$ and $r \times s$ matrices: $B = [b_{ik}]^{p,q}_{i,k=1}$ and $C = [c_{\sigma\tau}]^{r,s}_{\sigma,\tau=1}$, then $B \otimes C$ is defined as the following $(pr) \times (qs)$ matrix (Bellman, 1960; Marcus and Minc, 1964):

$$B \otimes C = [b_{ik}C]^{p,q}_{i,k=1} = [[b_{ik}c_{\sigma\tau}]^{r,s}_{\sigma,\tau=1}]^{p,q}_{i,k=1}. \tag{46}$$

Thus

$$A_{pj} = A_{w(p,i),w(j,i)}.$$

Letting $p = 1$ and noting (8), we have $A_{1j} = A_{i,w(j,i)}$ or

$$A_{1j} = \sum_{s=1}^{h} A_{is}\delta_{s,w(j,i)}.$$

Multiplying both sides by $\delta_{k,w(j,i)}$ and summing up with respect to j from 1 to h, we obtain

$$\sum_{j=1}^{h} A_{1j}\delta_{k,w(j,i)} = \sum_{s=1}^{h} A_{is} \sum_{j=1}^{h} \delta_{s,w(j,i)}\delta_{k,w(j,i)} = \sum_{s=1}^{h} A_{is}\delta_{sk} = A_{ik}.$$

Or, by virtue of (26),

$$A_{ik} = \sum_{j=1}^{h} q_{ik}(g_j)A_{1j}, \quad i, k = 1, \ldots, h. \tag{47}$$

Equation (45) follows from here in accordance with (46); we call it the Structural Formula.

□

Remark. According to the last eqn of (27), there is only one non-zero matrix term on the right side of (47) and this term has a scalar factor, say, $q_{ik}(g_{j_0})$ which is equal to 1. Thus eqns (45) and (47) mean that every block A_{ik} of matrix A_* is equal to one of its blocks located in the first block row: $A_{ik} = A_{1j_0}$, where $j_0 = j_0(i, k)$ is determined by the symmetry matrices $Q(g_j)$, $j = 1, \ldots, h$. Blocks $A_{1j}(j = 1, \ldots, h)$ will be called the basic blocks of A_* and the first subscript will be omitted: A_j, $j = 1, \ldots, h$.

5.2. Matrix A_* (44) may or may not be symmetric regardless of symmetry of the corresponding mechanical system. Matrix symmetry follows from reciprocity laws (relations) for corresponding quantities or equations, and any symmetric block matrix of order mh contains not more than $mh(mh+1)/2$ distinct elements. Symmetry of the mechanical system leads to the Structural Formula (45). According to (45), associated matrices have no more than m^2h different elements belonging to its basic blocks. If such a matrix is symmetric, then its basic blocks $A_1, A_2, \ldots, A_h$ are divided into two parts: blocks of one part are mutually transposed ($A_k = A_j^T$), while blocks of the other part are symmetric (self-transposed), and the total number of distinct elements reduces by half.

Matrices A_* (45) may be written explicitly. First let us replace the basic elements $b_{\mu+r}$ in the first four circulant matrices $B_r^{(k)}$ (10)–(13) by basic $m \times m$ blocks $A_{\mu+(v-1)n}$ of matrix A_* (44), $r \equiv (v-1)\,n = 0, n, 2n, 3n$:

$$A_r^{(1)} = \left[\begin{array}{c|c|c|c} A_{1+r} & A_{2+r} & \cdots & A_{n+r} \\ \hline A_{n+r} & A_{1+r} & \cdots & A_{n-1+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline A_{2+r} & A_{3+r} & \cdots & A_{1+r} \end{array}\right] \tag{48}$$

$$A_r^{(2)} = \left[\begin{array}{c|c|c|c} A_{1+r} & A_{n+r} & \cdots & A_{2+r} \\ \hline A_{2+r} & A_{1+r} & \cdots & A_{3+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline A_{n+r} & A_{n-1+r} & \cdots & A_{1+r} \end{array}\right] \tag{49}$$

$$A_r^{(3)} = \left[\begin{array}{c|c|c|c} A_{n+r} & A_{n-1+r} & \cdots & A_{1+r} \\ \hline A_{1+r} & A_{n+r} & \cdots & A_{2+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline A_{n-1+r} & A_{n-2+r} & \cdots & A_{n+r} \end{array}\right] \tag{50}$$

$$A_r^{(4)} = \left[\begin{array}{c|c|c|c} A_{n+r} & A_{1+r} & \cdots & A_{n-1+r} \\ \hline A_{n-1+r} & A_{n+r} & \cdots & A_{n-2+r} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline A_{1+r} & A_{2+r} & \cdots & A_{n+r} \end{array}\right] \tag{51}$$

$$r = 0, n, 2n, 3n.$$

Then, by substituting symmetry matrices $Q(g_i)$ of Table 4 into the Structural Formula, one can verify that matrices A_* (45) have the following configuration:

$$A_*(C_n) = [A_0^{(1)}]_{mn} \tag{52}$$

$$A_*(S_{2n}) = \left[\begin{array}{c|c} A_0^{(1)} & A_n^{(1)} \\ \hline A_n^{(4)} & A_0^{(1)} \end{array}\right]_{2mn} \tag{53}$$

$$A_*(C_{nh}) = \left[\begin{array}{c|c} A_0^{(1)} & A_n^{(1)} \\ \hline A_n^{(1)} & A_0^{(1)} \end{array}\right]_{2mn} \tag{54}$$

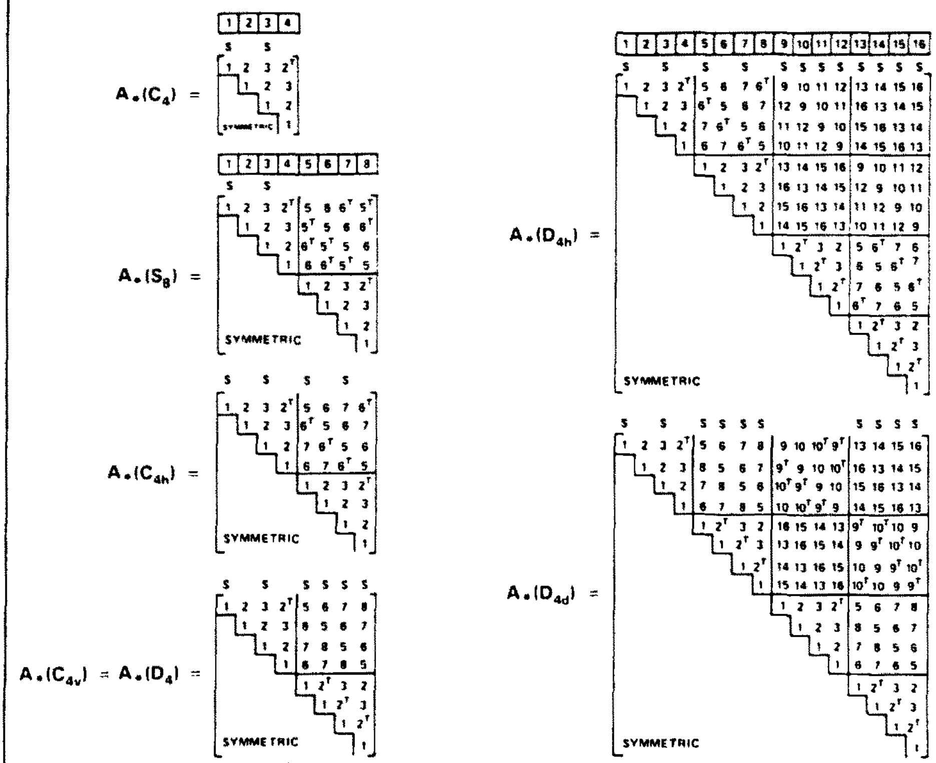
$$A_*(C_{nv}) = A_*(D_n) = \left[\begin{array}{c|c} A_0^{(1)} & A_n^{(1)} \\ \hline A_n^{(2)} & A_0^{(2)} \end{array}\right]_{2mn} \tag{55}$$

$$A_*(D_{nh}) = \left[\begin{array}{c|c|c|c} A_0^{(1)} & A_n^{(1)} & A_{2n}^{(1)} & A_{3n}^{(1)} \\ \hline A_n^{(1)} & A_0^{(1)} & A_{3n}^{(1)} & A_{2n}^{(1)} \\ \hline A_{2n}^{(2)} & A_{3n}^{(2)} & A_0^{(2)} & A_n^{(2)} \\ \hline A_{3n}^{(2)} & A_{2n}^{(2)} & A_n^{(2)} & A_0^{(2)} \end{array}\right]_{4mn} \tag{56}$$

$$A_*(D_{nd}) = \left[\begin{array}{c|c|c|c} A_0^{(1)} & A_n^{(1)} & A_{2n}^{(1)} & A_{3n}^{(1)} \\ \hline A_n^{(2)} & A_0^{(2)} & A_{3n}^{(3)} & A_{2n}^{(3)} \\ \hline A_{2n}^{(4)} & A_{3n}^{(4)} & A_0^{(1)} & A_n^{(1)} \\ \hline A_{3n}^{(2)} & A_{2n}^{(2)} & A_n^{(2)} & A_0^{(2)} \end{array}\right]_{4mn}. \tag{57}$$

More detailed symmetric matrices A_* are given in Table 6 for $n = 4$. Symbols S and T in this table state that the corresponding blocks are symmetric or transposed, respectively.

Table 6. Symmetric matrices A_* corresponding to the CSS, $n = 4$



NOTE IN ALL MATRICES INTEGER i STATES FOR BLOCK A_i

5.3. Consider the spectral properties of the Structural Formula (45):

Theorem 2. Matrices A_* (45) have the explicit block diagonal decomposition

$$A_* = U_* \Lambda_* U_*^{\mathrm{H}}, \tag{58}$$

where U_* is uninormal

$$U_* = U \otimes I_m = [u_{is} I_m]_{i,s=1}^{h}, \quad U_*^{\mathrm{H}} = U^{\mathrm{H}} \otimes I_m = [\bar{u}_{sk} I_m]_{s,k=1}^{h}, \tag{59}$$

Λ_* is block diagonal

$$\Lambda_* = \sum_{i=1}^{h} T(g_i) \otimes A_i, \tag{60}$$

and matrices $T(g_i)$, U and U^{H} are defined by (41)–(42), (35) *and* (37), *respectively.*

Proof. Kronecker's matrix product possesses the following property (Bellman, 1960; Marcus and Minc, 1964): if A, B and C are $p \times p$ matrices and F, G and H are $q \times q$ matrices, then

$$(ABC) \otimes (FGH) = (A \otimes F)(B \otimes G)(C \otimes H). \tag{61}$$

Hence, if matrix B is diagonal or block diagonal then matrix $B \otimes G$ is also block diagonal.

Therefore introducing (40) into (45) we obtain the block diagonal decomposition:

$$\begin{aligned} A_* &= \sum_{j=1}^{h} UT(g_j)U^{\mathrm{H}} \otimes (I_m A_j I_m) \\ &= (U \otimes I_m)\left(\sum_{j=1}^{h} T(g_j) \otimes A_j \right)(U^{\mathrm{H}} \otimes I_m) = U_* \Lambda_* U_*^{\mathrm{H}}. \end{aligned}$$

□

Remark. In accordance with (41), block diagonal matrix Λ_* has the configuration:

$$\Lambda_* = \begin{bmatrix} \underbrace{\Lambda_1 \quad \ddots \quad \Lambda_1}_{n_1 \text{ times}} & & & \\ & \underbrace{\Lambda_2 \quad \ddots \quad \Lambda_2}_{n_2 \text{ times}} & & \\ & & \ddots & \\ & & & \underbrace{\Lambda_H \quad \ddots \quad \Lambda_H}_{n_H \text{ times}} \end{bmatrix}, \qquad (62)$$

where

$$\Lambda_r = \sum_{j=1}^{h} \tau_r(g_j) \otimes A_j = \left[\sum_{j=1}^{h} \tau_{r\alpha\beta}(g_j) A_j \right]_{\alpha,\beta=1}^{n_r}, \quad r = 1, \ldots, H \qquad (63)$$

or

$$\Lambda_r = [\Lambda_{r\alpha\beta}]_{\alpha,\beta=1}^{n_r}, \quad \Lambda_{r\alpha\beta} = \sum_{j=1}^{h} \tau_{r\alpha\beta}(g_j) A_j, \quad r = 1, \ldots, H. \qquad (64)$$

Blocks Λ_r are of order mn_r, $m \leqslant mn_r \leqslant 5m$, and submatrices $\Lambda_{r\alpha\beta}$ have order m. In short, matrix Λ_* may be written as

$$\Lambda_* = [[\Lambda_r \delta_{\mu\nu} \delta_{rs}]_{\mu,\nu=1}^{n_r}]_{r,s=1}^{H} \qquad (65)$$

or even

$$\Lambda_* = [[[\Lambda_{r\alpha\beta} \delta_{\mu\nu} \delta_{rs}]_{\alpha,\beta=1}^{n_r}]_{\mu,\nu=1}^{n_r}]_{r,s=1}^{H}. \qquad (66)$$

Thus the majority of the FSS eigenvalues are at least of n_r-fold, $1 \leqslant n_r \leqslant 5$. In particular, the CSS possess double eigenfrequencies and critical loads.

With no loss in generality, we assume that matrices A_* (45) are real. Then if A_* is symmetric, the block diagonal matrix Λ_* is Hermitian: $A_* = U_* \Lambda_* U_*^{\mathrm{H}}$ and $A_* = A_*^{\mathrm{T}} = A_*^{\mathrm{H}} = U_* \Lambda_*^{\mathrm{H}} U_*^{\mathrm{H}}$, hence $\Lambda_*^{\mathrm{H}} = \Lambda_*$.

Consider now Theorem 3 which states that all operations with matrices corresponding to the same FSS preserve the Structural Formula (45).

Theorem 3. *If* $A_* = \Sigma_{j=1}^h Q(g_j) \otimes A_j$ *and* $B_* = \Sigma_{j=1}^h Q(g_j) \otimes B_j$, *then*

$$A_* + B_* = \sum_{j=1}^{h} Q(g_j) \otimes (A_j + B_j) \tag{67}$$

$$A_* B_* = \sum_{j=1}^{h} Q(g_j) \otimes C_j = C_*. \tag{68}$$

$$A_*^{-1} = \sum_{j=1}^{h} Q(g_j) \otimes \tilde{A}_j. \tag{69}$$

Proof. In accordance with (45) and (47),

$$A_* = \sum_{j=1}^{h} [q_{ik}(g_j) A_j]_{i,k=1}^h, \quad B_* = \sum_{j=1}^{h} [q_{ik}(g_j) B_j]_{i,k=1}^h.$$

Hence,

$$A_* + B_* = \sum_{j=1}^{h} [q_{ik}(g_j)(A_j + B_j)]_{i,k=1}^h = \sum_{j=1}^{h} Q(g_j) \otimes (A_j + B_j).$$

Next, we consider a matrix product. Invoking (38), we have

$$\begin{aligned} C_* = A_* B_* &= \left(\sum_{s=1}^{h} Q(g_s) \otimes A_s \right) \left(\sum_{t=1}^{h} Q(g_t) \otimes B_t \right) \\ &= \sum_{s,t=1}^{h} Q(g_s) Q(g_t) \otimes A_s B_t = \sum_{s,t=1}^{h} Q(g_{w(t,s)}) \otimes A_s B_t \\ &= \sum_{s=1}^{h} \left(\sum_{t=1}^{h} Q(g_{w(t,s)}) \otimes A_s B_t \right) = \sum_{s=1}^{h} C_*^{(s)} \end{aligned}$$

where

$$C_*^{(s)} = \sum_{t=1}^{h} Q(g_{w(t,s)}) \otimes A_s B_t.$$

Thus $C_*^{(s)}$ is described by the Structural Formula. Since a sum of such matrices also possesses this formula, eqn (68) is proven. Let us find the basic blocks C_j of $C_* = A_* B_*$. Denote blocks of $C_*^{(s)}$ by $C_{ij}^{(s)}, i, j = 1, \ldots, h$. Then its basic blocks are

$$C_j^{(s)} \equiv C_{1j}^{(s)} = \sum_{t=1}^{h} q_{1j}(g_{w(t,s)}) A_s B_t = \sum_{t=1}^{h} \delta_{j,w(w(t,s),1)} A_s B_t, \quad j = 1, \ldots, h.$$

Since $w(w(t, s), 1) = w(t, w(s, 1)) = w(t, s)$, we obtain

$$C_j^{(s)} = \sum_{t=1}^{h} \delta_{j,w(t,s)} A_s B_t, \quad j = 1, \ldots, h.$$

A single non-zero term in the right side is determined by $w(t, s) = j$, i.e. by $g_t g_s = g_j$, which means that $t = t(j, s)$ and therefore

$$C_j^{(s)} = A_s B_{j(s)}, \quad j = 1, \ldots, h$$

where $j(s)$ is defined by the matrix G_w. Hence, the basic blocks of matrix product $C_* = A_* B_*$

(68) are

$$C_j = \sum_{\nu=1}^{h} C_j^{(\nu)} = \sum_{\nu=1}^{h} A_\nu B_{l(\nu)}, \quad j = 1, \ldots, h. \tag{70}$$

Finally, the existence of eqn (69) simply follows from the Structural Formula for non-singular matrix A_*, and our goal is to compute basic blocks $\tilde{A}_j$, $j = 1, \ldots, h$, of the inverse matrix A_*^{-1}. To this end, matrix A_*^{-1} is presented as a triple product $A_*^{-1} = U_* \Lambda_*^{-1} U_*^{H}$ and in accordance with (60) and (65),

$$\Lambda_*^{-1} = \sum_{j=1}^{h} T(g_j) \otimes \tilde{A}_j = [[\Lambda_r^{-1} \delta_{\mu\nu} \delta_{rs}]_{\mu,\nu=1}^{n_r}]_{r,s=1}^{H}. \tag{71}$$

Assuming that blocks Λ_r^{-1}, $r = 1, \ldots, H$ are found numerically by inversion of blocks Λ_r (63), we present them in the partitioned form (64)

$$\Lambda_r^{-1} = [\tilde{\Lambda}_{r\alpha\beta}]_{\alpha,\beta=1}^{n_r} \tag{72}$$

and propose that similar to $\Lambda_{r\alpha\beta}$ (64),

$$\tilde{\Lambda}_{r\alpha\beta} = \sum_{j=1}^{h} \tau_{r\alpha\beta}(g_j) \tilde{A}_j, \quad \alpha, \beta = 1, \ldots, n_r; \quad r = 1, \ldots, H. \tag{73}$$

Then multiplying both sides of (73) by $(n_r/h)\bar{\tau}_{r\alpha\beta}(g_k)$, summing up with respect to α and β from 1 to n_r and r from 1 to H, and taking into account the orthogonality relation (36), we obtain

$$\sum_{r=1}^{H} \frac{n_r}{h} \sum_{\alpha,\beta=1}^{n_r} \tilde{\Lambda}_{r\alpha\beta} \bar{\tau}_{r\alpha\beta}(g_k) = \sum_{j=1}^{h} \tilde{A}_j \sum_{r=1}^{H} \frac{n_r}{h} \sum_{\alpha,\beta=1}^{n_r} \tau_{r\alpha\beta}(g_j) \bar{\tau}_{r\alpha\beta}(g_k) = \sum_{j=1}^{h} \tilde{A}_j \delta_{jk} = \tilde{A}_k.$$

Thus

$$\tilde{A}_j = \sum_{r=1}^{H} \frac{n_r}{h} \sum_{\alpha,\beta=1}^{n_r} \tilde{\Lambda}_{r\alpha\beta} \bar{\tau}_{r\alpha\beta}(g_j), \quad j = 1, \ldots, h. \tag{74}$$

□

Remark 1. Basic blocks C_j (70) may be written explicitly:

$$C_{\mu+(\nu-1)n} = \sum_{\mu_1=1}^{n} \sum_{\nu_1=1}^{p=h/n} A_{\mu_1+(\nu_1-1)n} B_{m_{\nu_1}}, \tag{75}$$

$$\mu = 1, \ldots, n; \quad \nu = 1, \ldots, p,$$

subscripts m_{ν_1}, which depend on ν, are depicted in Table 7 for all CSS.

Remark 2. Equations (68), (75) and (69), (74) are very efficient in computing the radiation matrices (heat transfer problems).

6. SOLUTION OF LINEAR EQUATIONS INTRODUCED ON THE FSS AND SYMMETRY OF THE APPLIED LOADS

6.1. The explicit block diagonal decomposition of matrix A_* (45) given by Theorem 2 leads to substantial simplifications in the solving of linear equations corresponding to the FSS.

Table 7. Basic blocks $C_{\mu+(\nu-1)n}$ of matrix $C_* = A_* B_*$

$$C_{\mu+(\nu-1)n} = \sum_{\mu_1=1}^{n} \sum_{\nu_1=1}^{p=h/n} A_{\mu_1+(\nu_1-1)n} B_{m_{\nu_1}} \qquad \mu = 1,\ldots,n; \quad \nu = 1,\ldots,p$$

GROUP	p	m_{ν_1}	C_μ	$C_{\mu+n}$	$C_{\mu+2n}$	$C_{\mu+3n}$	COMMENTS
C_n	1	$m_1 =$	i_1				
S_{2n}	2	$m_1 =$	i_1	$i_1 + n$			$i_1 = \begin{cases} \mu-\mu_1+1, & \text{IF } \mu-\mu_1+1>0 \\ \mu-\mu_1+1+n, & \text{OTHERWISE} \end{cases}$
		$m_2 =$	$i_3 + n$	i_1			
C_{nh}	2	$m_1 =$	i_1	$i_1 + n$			
		$m_2 =$	$i_1 + n$	i_1			
C_{nv} AND D_n	2	$m_1 =$	i_1	$i_1 + n$			$i_2 = \begin{cases} \mu_1-\mu+1, & \text{IF } \mu_1-\mu+1>0 \\ \mu_1-\mu+1+n, & \text{OTHERWISE} \end{cases}$
		$m_2 =$	$i_2 + n$	i_2			
D_{nh}	4	$m_1 =$	i_1	$i_1 + n$	$i_1 + 2n$	$i_1 + 3n$	
		$m_2 =$	$i_1 + n$	i_1	$i_1 + 3n$	$i_1 + 2n$	
		$m_3 =$	$i_2 + 2n$	$i_2 + 3n$	i_2	$i_2 + n$	$i_3 = \begin{cases} \mu-\mu_1, & \text{IF } \mu-\mu_1>0 \\ \mu-\mu_1+n, & \text{OTHERWISE} \end{cases}$
		$m_4 =$	$i_2 + 3n$	$i_2 + 2n$	$i_2 + n$	i_2	
D_{nd}	4	$m_1 =$	i_1	$i_1 + n$	$i_1 + 2n$	$i_1 + 3n$	
		$m_2 =$	$i_2 + n$	i_2	$i_4 + 3n$	$i_4 + 2n$	$i_4 = \begin{cases} \mu_1-\mu, & \text{IF } \mu_1-\mu>0 \\ \mu_1-\mu+n, & \text{OTHERWISE} \end{cases}$
		$m_3 =$	$i_3 + 2n$	$i_3 + 3n$	i_1	$i_1 + n$	
		$m_4 =$	$i_2 + 3n$	$i_2 + 2n$	$i_2 + n$	i_2	

Theorem 4. *A system of mh linear equations*

$$A_* x_* = b_*, \quad x_* = [x_i]_{i=1}^h, \quad b_* = [b_k]_{k=1}^h, \tag{76}$$

defined on the FSS with symmetry group G, is divided into H uncoupled subsystems of order mn_r $(1 \leqslant n_r \leqslant 5)$, *each containing* n_r *unknown subvectors* $y_{r\gamma}$

$$\Lambda_r[y_{r1}, \ldots, y_{rn_r}] = [c_{r1}, \ldots, c_{rn_r}], \quad r = 1, \ldots, H \tag{77}$$

where Λ_r *is determined by* (63)–(64) *and*

$$c_{r\gamma} = (n_r/h)^{1/2} \left[\sum_{k=1}^{h} b_k \bar{\tau}_{r\gamma\beta}(g_k) \right]_{\beta=1}^{n_r}, \quad \gamma = 1, \ldots, n_r; \quad r = 1, \ldots, H. \tag{78}$$

Once subsystems (77) *are solved, the initial unknowns are found from*

$$x_i = \sum_{r=1}^{H} \left(\frac{n_r}{h} \right)^{1/2} \sum_{\gamma,\alpha=1}^{n_r} y_{r\gamma\alpha} \tau_{r\gamma\alpha}(g_i), \quad i = 1, \ldots, h \tag{79}$$

where $y_{r\gamma\alpha}$ *belong to subvector* $y_{r\gamma} = [y_{r\gamma\alpha}]_{\alpha=1}^{n_r}, \quad \gamma = 1, \ldots, n_r.$

Proof. Substitute (58) into (76) and then premultiply the result by U_*^{H} (59):

$$\Lambda_* U_*^{\mathrm{H}} x_* = U_*^{\mathrm{H}} b_*$$

or

$$\Lambda_* y_* = c_*, \tag{80}$$

where

$$c_* = U_*^{\mathrm{H}} b_*, \quad c_* = [[c_{r_j}]_{j=1}^{n_r}]_{r=1}^{H}. \tag{81}$$

Subvectors c_{r_j} are calculated by (78) in accordance with (37) and (59), and

$$y_* = U_*^{\mathrm{H}} x_*, \quad y_* = [[y_{r_j}]_{j=1}^{n_r}]_{r=1}^{H}. \tag{82}$$

Written in full, system (80) has the form

$$\begin{bmatrix} \underbrace{\Lambda_1 \ \ddots \ \Lambda_1}_{n_1 \text{ times}} & & & \\ & \underbrace{\Lambda_2 \ \ddots \ \Lambda_2}_{n_2 \text{ times}} & & \\ & & \ddots & \\ & & & \underbrace{\Lambda_H \ \ddots \ \Lambda_H}_{n_H \text{ times}} \end{bmatrix} \begin{bmatrix} y_{11} \\ \cdots \\ y_{1n_1} \\ \hline y_{21} \\ \cdots \\ y_{2n_2} \\ \hline \ddots \\ \hline y_{H1} \\ \cdots \\ y_{Hn_H} \end{bmatrix} = \begin{bmatrix} c_{11} \\ \cdots \\ c_{1n_1} \\ \hline c_{21} \\ \cdots \\ c_{2n_2} \\ \hline \ddots \\ \hline c_{H1} \\ \cdots \\ c_{Hn_H} \end{bmatrix} \tag{83}$$

and may be presented as (77). Having obtained solutions of subsystems (77), one shall compute vector $x_* = U_* y_*$ whose subvectors are determined by (79) due to eqns (35) and (59).

□

To estimate the efficiency of this method we compare it with Gaussian elimination, the standard widely used procedure. To simplify the problem, we assume that all of the matrices under consideration are of a full scale. Denote respectively by M_1^{G} and M_2^{G} the number of operations (multiplications) and storage requirements for Gaussian elimination, and by M_1^{S} and M_2^{S} the analogous quantities corresponding to the symmetry approach. Then

$$M_1^{\mathrm{G}} \sim (mh)^3, \quad M_2^{\mathrm{G}} \sim (mh)^2,$$

while

$$M_1^{\mathrm{S}} \sim m^3 h, \quad M_2^{\mathrm{S}} \sim m^2 h.$$

Hence the efficiency of the symmetry approach can be described by two characteristics

$$N_1 = M_1^{\mathrm{G}}/M_1^{\mathrm{S}} \sim h^2 \quad \text{and} \quad N_2 = M_2^{\mathrm{G}}/M_2^{\mathrm{S}} \sim h. \tag{84}$$

6.2. The efficiency of the symmetry approach depends on symmetry of the applied load. Although hitherto nothing was said on this, some assumptions have already been made by default. To reveal them we divide the whole set of the applied loads into two subsets (which can intersect): the active and the parametric loads. We define them with respect to their locations in matrix equation (76). The active loads are located on the right side; they form the load vector b_*. The parametric loads occupy the left side of (76); they participate in the formation of matrices, hence, they are inertia, damping, gyroscopic, and circulatory forces, thermal loads, etc. Some of them such as thermal loads, for instance, may be parametric and active simultaneously. In accordance with Theorem 4, matrix A_* of eqn (76) corresponds to the FSS with group G. This group describes symmetry of the system, i.e. symmetry of its geometry (including supports) and materials. Since matrix A_* also depends on the parametric load, it was implied tacitly that the parametric load is symmetric too. Moreover, it was assumed that its symmetry is identical with the symmetry of the unloaded system. Evidently, this is not necessary. Let us denote symmetry of the unloaded system by S_1 and symmetry of the applied parametric load (loads) by S_2. Then the actual (total) symmetry of the loaded system is defined as the intersection of S_1 and S_2

$$S = S_1 \cap S_2. \tag{85}$$

Hence, the loaded system will be symmetric if S_1 and S_2 have common symmetry elements: axes and/or planes. It is useful to note that the intersection of two congruent axes c_{n_1} of S_1 and c_{n_2} of S_2 is not necessarily a symmetry element because axis $c_n = c_{n_2} \cap c_{n_1}$ may have order one. Axis c_n will be a symmetry element of S (85) if $n = \gcd(n_1, n_2) > 1$ where "gcd" means greatest common divisor. Let G_1, G_2 and G be symmetry groups corresponding to S_1, S_2 and S, respectively, and h_1, h_2 and h be their orders. Then G is a common subgroup of G_1 and G_2, not necessarily the largest, and $h \leqslant \gcd(h_1, h_2)$. It is convenient to treat S as a composite system containing two subsystems: a real S_1 and an imaginary S_2 enclosed into S_1. Then each primitive of S consists of h_1/h primitives of S_1 and of h_2/h primitives of S_2. In fact, the primitives of S are

$$S_j = (h_1/h)S^{(1)}_{j_1}, \quad j = 1, \ldots, h; \quad j_1 = 1, \ldots, h_1. \tag{86}$$

They are subjected to the parametric load distributed along S_j in accordance with $(h_2/h)S^{(2)}_{j_2}$. All of the FSS composed of two CSS, one within the other, are shown in Table 8.

Table 8. The highest symmetry of systems composed of two CSS

S_1 \ S_2	C_{n_2}	S_{2n_2}	C_{n_2h}	C_{n_2v}	D_{n_2}	D_{n_2h}	D_{n_2d}
C_{n_1}	C_n *	C_n	C_n	C_n	C_n	C_n	C_n
S_{2n_1}		S_{2n}	C_n	C_n	C_n	C_n	S_{2n}
C_{n_1h}			C_{nh}	C_n	C_n	C_{nh}	C_n
C_{n_1v}				C_{nv}	C_n	C_{nv}	C_{nv}
D_{n_1}					D_n	D_n	D_n
D_{n_1h}	SYMMETRIC					D_{nh}	C_{nv}
D_{n_1d}							D_{nd}

* $n = \gcd(n_1, n_2)$

6.3. Theorem 4 imposes no restrictions on the load vector b_* (76), hence the efficiency of the symmetry approach given by (84) is obtained for a certain nonsymmetric active load. However, if the active load is symmetric and its symmetry is identical with the actual symmetry of the FSS, the load vector b_* consists of h identical subvectors

$$b_1 = \cdots = b_h = b_0. \tag{87}$$

Substitute (87) into (78):

$$c_{r\gamma} = (n_r/h)^{1/2}\left(\sum_{k=1}^{h} \bar{\tau}_{r\gamma\beta}(g_k)\right)b_0. \tag{88}$$

Assume that the actual symmetry of the FSS is, for example, C_n. Then one can find from Table 5 that group C_n has $h = n$ unitary irreducible representations τ_r with elements

$$\tau_{r\gamma\beta}(g_k) \equiv \tau_{r11}(g_k) \equiv \tau_r(g_k) = \varepsilon_{k-1}^r = \exp(ir(k-1)2\pi/n), \quad i = \sqrt{-1}$$
$$r = 1, \ldots, H = h = n.$$

Since

$$\sum_{k=1}^{h} \bar{\tau}_{r\gamma\beta}(g_k) = \sum_{k=1}^{n} \exp(-ir(k-1)2\pi/n) = n\delta_{rn}$$

we have in this case

$$c_{r\gamma} \equiv c_{r1} \equiv c_r = \sqrt{h}\, b_0\delta_{rn}, \quad r = 1, \ldots, H = h = n.$$

Hence, $y_1 = \cdots = y_{n-1} = 0$ while y_n is determined from

$$\Lambda_n y_n = c_n, \quad \text{where} \quad \Lambda_n = \sum_{j=1}^{h} A_j \quad \text{and} \quad c_n = \sqrt{h}b_0. \tag{89}$$

It follows from here that vector $x_* = U_* y_*$ also consists of $h = n$ identical subvectors

$$x_1 = \cdots = x_h = (1/\sqrt{h})y_n. \tag{90}$$

Since it was required to solve only one subsystem (77) and it is of order m, the efficiency of the symmetry approach increases h times:

$$N_1 \sim h^3 \quad \text{and} \quad N_2 \sim h^2. \tag{91}$$

This result may be generalized. Let us present the load vector b_* defined by (87) as the following Kronecker product

$$b_{*1} \equiv b_* = v_1 \otimes b_0 = \begin{bmatrix} b_0 \\ \cdots \\ b_0 \end{bmatrix}_{mh}, \tag{92}$$

where

$$v_1^T = [1, 1, \ldots, 1]_h. \tag{93}$$

Each group has a one-dimensional irreducible representation consisting of h positive units: $\tau_u(g_j) = 1, j = 1, \ldots, n$. It is called the unity representation. Observation of Table 5 shows that the unity representation of groups C_n and C_{nh} is τ_n, for group S_{2n}, $\tau_u(g_i) = \tau_{2n}(g_i)$ since $\varepsilon^{2n}_{\mu-1/2} = 1$, regardless of μ. Finally τ_1 is the unity representation of groups C_{nv}, D_n, D_{nh} and D_{nd}. Clearly τ_u can be given by v_1 (93). Hence eqn (92) simply means that the active load follows the unity representation: if the active load has a symmetry group G, then it follows the unity representation of this group, and vice versa. The most common case in symmetry of the active load is that in which the load follows an irreducible representation of a certain subgroup of group G describing symmetry of the loaded system. Such a case will be studied in the next section. Now, for the sake of simplicity, we will consider a case of the active loads following the irreducible representations of a particular symmetry group D_{4h}, and we assume that the actual symmetry of the FSS is also D_{4h}. Such a system is shown in Fig. 1. It possesses four vertical $(\sigma_v^{(1)}, \ldots, \sigma_v^{(4)})$ and one horizontal (σ_h) planes of reflection, the vertical principal axis c_4 and four horizontal axes $c_2^{(1)}, \ldots, c_2^{(4)}$. System D_{4h} contains $h = 4n = 16$ primitives.

Let A_* be the stiffness matrix of D_{4h}; then it is real, symmetric and has the following configuration:

$$A_*(D_{4h}) = \begin{array}{cccc|cccc|cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
A_1 & & & & A_5 & & & & A_9 & & & A_{12} & A_{13} & & & A_{16} \\
 & A_1 & & & & A_5 & & & A_{12} & A_9 & & & A_{16} & A_{13} & & \\
 & & A_1 & & & & A_5 & & & A_{12} & A_9 & & & A_{16} & A_{13} & \\
 & & & A_1 & & & & A_5 & & & A_{12} & A_9 & & & A_{16} & A_{13} \\
\hline
 & & & & A_1 & & & & A_{13} & & & A_{16} & A_9 & & & A_{12} \\
 & & & & & A_1 & & & A_{16} & A_{13} & & & A_{12} & A_9 & & \\
 & & & & & & A_1 & & & A_{16} & A_{13} & & & A_{12} & A_9 & \\
 & & & & & & & A_1 & & & A_{16} & A_{13} & & & A_{12} & A_9 \\
\hline
 & & & & & & & & A_1 & & & & A_5 & & & \\
 & & & & & & & & & A_1 & & & & A_5 & & \\
 & & & & & & & & & & A_1 & & & & A_5 & \\
 & & & & & & & & & & & A_1 & & & & A_5 \\
\hline
 & & & & & & & & & & & & A_1 & & & \\
 & & & & & & & & & & & & & A_1 & & \\
 & & & & & & & & & & & & & & A_1 & \\
\text{SYMMETRIC} & & & & & & & & & & & & & & & A_1
\end{array} \quad . \qquad (94)$$

It is completely determined by six non-zero $m \times m$ basic blocks A_1, A_5, A_9, A_{12}, A_{13} and A_{16}, all of which are symmetric in accordance with Table 6.

One can find from Table 5 that group D_{4h} has eight one-dimensional irreducible representations $\tau_1, \ldots, \tau_8$ and two two-dimensional ones, τ_9 and τ_{10}; see also Table 9. Using them we build matrix U (35) whose construction was described in Section 4. It is given in Table 10. Now, if the active load follows the one-dimensional representation τ_r, then the load vector b_* has the form†

$$b_{*r} = u_r \otimes b, \quad 1 \leqslant r \leqslant 8 \qquad (95)$$

where u_r is the r-th column of U. If $\dim \tau_r (= n_r) > 1$, then there are n_r^2 active loads which

† Notice that $b = (n_1/h)^{-1/2} b_0$, where b_0 corresponds to eqn (92). It follows from $b_{*1} = u_1 \otimes b = (n_1/h)^{1/2} v_1 \otimes b$ and $b_{*1} = v_1 \otimes b_0$.

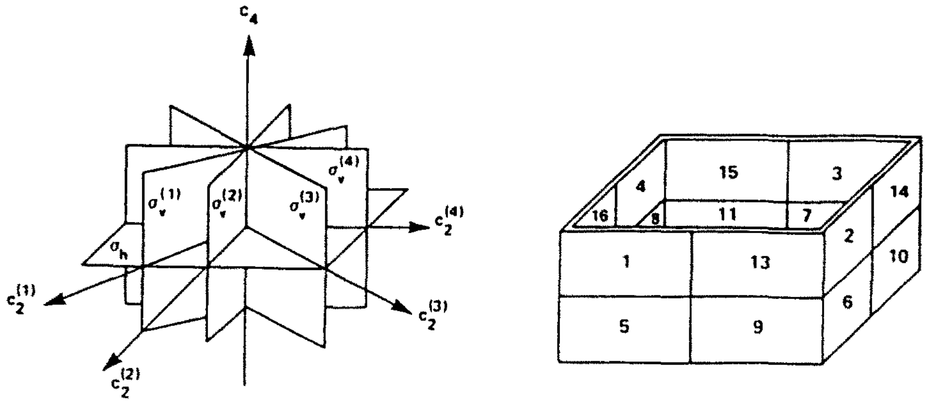


Fig. 1. System D_{4h}.

follow τ_r. For example, two-dimensional τ_9 is presented in matrix $U(D_{4h})$ by four columns, namely, $u_9 = u_{9,11}$, $u_{10} = u_{9,12}$, $u_{11} = u_{9,21}$, and $u_{12} = u_{9,22}$; see Table 10. Hence there are four active loads following τ_9:

$$h_{*9} = u_9 \otimes h, \quad h_{*10} = u_{10} \otimes h, \quad h_{*11} = u_{11} \otimes h, \quad h_{*12} = u_{12} \otimes h. \tag{96}$$

In fact all of these loads are very similar: $h_{*12} = \bar{h}_{*9}$, $h_{*11} = \bar{h}_{*10}$ and h_{*10} may be obtained from h_{*9} by rotation of the active load about the principal axis c_4 through 90° counterclockwise. (Notice that loads which follow complex irreducible representations are

Table 9. Irreducible representations of group D_{4h}

IRREDUCIBLE REPRESENTATIONS		ELEMENTS AND MATRICES OF IRREDUCIBLE REPRESENTATIONS τ_r			
		$\tau_r(g_\mu)$ $\mu = 1,2,3,4$	$\tau_r(g_{\mu+4})$ $\mu = 1,2,3,4$	$\tau_r(g_{\mu+8})$ $\mu = 1,2,3,4$	$\tau_r(g_{\mu+12})$ $\mu = 1,2,3,4$
ONE-DIMENSIONAL	τ_1	1	1	1	1
	τ_2	1	1	−1	−1
	τ_3	1	−1	1	−1
	τ_4	1	−1	−1	1
	τ_5	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$
	τ_6	$(-1)^{\mu-1}$	$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu}$
	τ_7	$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu-1}$	$(-1)^{\mu}$
	τ_8	$(-1)^{\mu-1}$	$(-1)^{\mu}$	$(-1)^{\mu}$	$(-1)^{\mu-1}$
TWO-DIMENSIONAL	τ_9	$\begin{bmatrix} \epsilon_{\mu-1} & \\ & \bar{\epsilon}_{\mu-1} \end{bmatrix}$	$\begin{bmatrix} \epsilon_{\mu-1} & \\ & \bar{\epsilon}_{\mu-1} \end{bmatrix}$	$\begin{bmatrix} & \epsilon_{\mu-1} \\ \bar{\epsilon}_{\mu-1} & \end{bmatrix}$	$\begin{bmatrix} & \epsilon_{\mu-1} \\ \bar{\epsilon}_{\mu-1} & \end{bmatrix}$
	τ_{10}	$\begin{bmatrix} \epsilon_{\mu-1} & \\ & \bar{\epsilon}_{\mu-1} \end{bmatrix}$	$\begin{bmatrix} -\epsilon_{\mu-1} & \\ & -\bar{\epsilon}_{\mu-1} \end{bmatrix}$	$\begin{bmatrix} & -\epsilon_{\mu-1} \\ -\bar{\epsilon}_{\mu-1} & \end{bmatrix}$	$\begin{bmatrix} & \epsilon_{\mu-1} \\ \bar{\epsilon}_{\mu-1} & \end{bmatrix}$

NOTE: $\epsilon_\mu = \exp(i\mu\pi/2)$, $i = \sqrt{-1}$

Table 10. Matrix $U(D_{4h})$

μ	i		ONE DIMENSIONAL IRR REPRESENTATIONS								TWO DIMENSIONAL IRREDUCIBLE REPRESENTATIONS							
			r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9				r_{10}			
			$r_{111}(\theta_i)$	$r_{211}(\theta_i)$	$r_{311}(\theta_i)$	$r_{411}(\theta_i)$	$r_{511}(\theta_i)$	$r_{611}(\theta_i)$	$r_{711}(\theta_i)$	$r_{811}(\theta_i)$	$r_{911}(\theta_i)$	$r_{912}(\theta_i)$	$r_{921}(\theta_i)$	$r_{922}(\theta_i)$	$r_{1011}(\theta_i)$	$r_{1012}(\theta_i)$	$r_{1021}(\theta_i)$	$r_{1022}(\theta_i)$
			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	$i=\mu$	1	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$
2		2	1/4	1/4	1/4	1/4	−1/4	−1/4	−1/4	−1/4	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$
3		3	1/4	1/4	1/4	1/4	1/4	1/4	1/4	1/4	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$
4		4	1/4	1/4	1/4	1/4	−1/4	−1/4	−1/4	−1/4	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$
1	$i=\mu+4$	5	1/4	1/4	−1/4	−1/4	1/4	1/4	−1/4	−1/4	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$
2		6	1/4	1/4	−1/4	−1/4	−1/4	−1/4	1/4	1/4	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$
3		7	1/4	1/4	−1/4	−1/4	1/4	1/4	−1/4	−1/4	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$
4		8	1/4	1/4	−1/4	−1/4	−1/4	−1/4	1/4	1/4	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$
1	$i=\mu+8$	9	1/4	−1/4	1/4	−1/4	1/4	−1/4	1/4	−1/4		$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	
2		10	1/4	−1/4	1/4	−1/4	−1/4	1/4	−1/4	1/4		$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	
3		11	1/4	−1/4	1/4	−1/4	1/4	−1/4	1/4	−1/4		$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	
4		12	1/4	−1/4	1/4	−1/4	−1/4	1/4	−1/4	1/4		$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	
1	$i=\mu+12$	13	1/4	−1/4	−1/4	1/4	1/4	−1/4	−1/4	1/4		$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	
2		14	1/4	−1/4	−1/4	1/4	−1/4	1/4	1/4	−1/4		$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	
3		15	1/4	−1/4	−1/4	1/4	1/4	−1/4	−1/4	1/4		$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	
4		16	1/4	−1/4	−1/4	1/4	−1/4	1/4	1/4	−1/4		$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$			$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	

not the standard mechanical loads.) The total number of active loads which can be associated with all the irreducible representations of D_{4h} is $h = 16$. These loads form a $16m \times 16$ matrix

$$B_* = [b_{*1} b_{*2} \ldots b_{*16}] = U \otimes b. \tag{97}$$

The corresponding system of linear equations

$$A_* X_* = B_* \tag{98}$$

is decomposed to

$$\Lambda_* Y_* = C_*$$

and the $16m \times 16$ matrix C_*, which in accordance with (81) is equal to $U_*^{H} B_*$, is of the following form

$$\begin{aligned} C_* &= U_*^{H} B_* = (U^{H} \otimes I_m)(U \otimes b) \\ &= (U^{H} U) \otimes (I_m b) = I_h \otimes b = I_{16} \otimes b, \end{aligned} \tag{99}$$

that is,

$$C_* = \begin{bmatrix} b & & & \\ & b & & \\ & & \ddots & \\ & & & b \end{bmatrix}_{16m \times 16} = [c_{*1} c_{*2} \ldots c_{*16}]. \tag{100}$$

To find the configuration of vector x_* we first need to study matrix Λ_*. It has the form:

$$\Lambda_* = \begin{bmatrix} \Lambda_1 & & & & & & & & & & & \\ & \Lambda_2 & & & & & & & & & & \\ & & \Lambda_3 & & & & & & & & & \\ & & & \Lambda_4 & & & & & & & & \\ & & & & \Lambda_5 & & & & & & & \\ & & & & & \Lambda_6 & & & & & & \\ & & & & & & \Lambda_7 & & & & & \\ & & & & & & & \Lambda_8 & & & & \\ & & & & & & & & \Lambda_9 & & & \\ & & & & & & & & & \Lambda_9 & & \\ & & & & & & & & & & \Lambda_{10} & \\ & & & & & & & & & & & \Lambda_{10} \end{bmatrix} \tag{101}$$

and according to (63) and (6),

$$\Lambda_r = \sum_{j=1}^{h=16} \tau_r(g_j) \otimes A_j = \sum_{\mu=1}^{n=4} \sum_{\nu=1}^{p=4} \tau_r(g_{\mu+4(\nu-1)}) \otimes A_{\mu+4(\nu-1)} \tag{102}$$

$$r = 1, \ldots, H = 10.$$

Blocks Λ_1–Λ_8 are of order m; we present them in the form

$$\begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \ldots \\ \Lambda_8 \end{bmatrix} = \sum_{\mu=1}^{n=4} (S_\mu \otimes I_m) \begin{bmatrix} A_\mu \\ A_{\mu+4} \\ A_{\mu+8} \\ A_{\mu+12} \end{bmatrix} \tag{103}$$

where matrices S_μ, in accordance with Table 9, are the following:

$$S_\mu = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ (-1)^{\mu-1} & (-1)^{\mu-1} & (-1)^{\mu-1} & (-1)^{\mu-1} \\ (-1)^{\mu-1} & (-1)^{\mu-1} & (-1)^{\mu} & (-1)^{\mu} \\ (-1)^{\mu-1} & (-1)^{\mu} & (-1)^{\mu-1} & (-1)^{\mu} \\ (-1)^{\mu-1} & (-1)^{\mu} & (-1)^{\mu} & (-1)^{\mu-1} \end{bmatrix} \tag{104}$$

$$\mu = 1, 2, 3, 4.$$

Blocks Λ_9 and Λ_{10} (of order $2m$) are

$$\Lambda_9 = \sum_{\mu=1}^{n=4} \left[\begin{array}{c|c} \varepsilon_{\mu-1}(A_\mu + A_{\mu+4}) & \varepsilon_{\mu-1}(A_{\mu+8} + A_{\mu+12}) \\ \hline \bar{\varepsilon}_{\mu-1}(A_{\mu+8} + A_{\mu+12}) & \bar{\varepsilon}_{\mu-1}(A_\mu + A_{\mu+4}) \end{array}\right] \tag{105}$$

$$\Lambda_{10} = \sum_{\mu=1}^{n=4} \left[\begin{array}{c|c} \varepsilon_{\mu-1}(A_\mu - A_{\mu+4}) & -\varepsilon_{\mu-1}(A_{\mu+8} - A_{\mu+12}) \\ \hline -\bar{\varepsilon}_{\mu-1}(A_{\mu+8} - A_{\mu+12}) & \bar{\varepsilon}_{\mu-1}(A_\mu - A_{\mu+4}) \end{array}\right]. \tag{106}$$

Matrix A_* is real, hence

$$\Lambda_9 = \left[\begin{array}{c|c} \Lambda_{9,11} & \Lambda_{9,12} \\ \hline \bar{\Lambda}_{9,12} & \bar{\Lambda}_{9,11} \end{array}\right] \text{ and } \Lambda_{10} = \left[\begin{array}{c|c} \Lambda_{10,11} & \Lambda_{10,12} \\ \hline \bar{\Lambda}_{10,12} & \bar{\Lambda}_{10,11} \end{array}\right]. \tag{107}$$

Now we assume that the active load follows a one-dimensional representation τ_r, $1 \leqslant r \leqslant 8$. Then one has to solve the $m \times m$ subsystem

$$\Lambda_r y_r = b, \quad 1 \leqslant r \leqslant 8. \tag{108}$$

The entire vector y_{*r} (whose order is $mh = 16m$) is determined by

$$y_{*r} = i_r \otimes y_r, \quad 1 \leqslant r \leqslant 8 \tag{109}$$

where i_r is the r-th column of the identity matrix $I_h = I_{16}$. Therefore, vector x_* of the initial unknowns is

$$\begin{aligned} x_{*r} &= U_* y_{*r} = (U \otimes I_m)(i_r \otimes y_r) \\ &= (Ui_r) \otimes (I_m y_r) = u_r \otimes y_r, \quad 1 \leqslant r \leqslant 8. \end{aligned} \tag{110}$$

Clearly, its subvectors $x_{1r}, \ldots, x_{hr}$ are distinguished only in scalar factors ± 1. The efficiency given by eqn (91) follows from here. This result may be formulated as:

Theorem 5. If the active load follows any one-dimensional irreducible representation of group G describing the actual symmetry of the FSS, then the efficiency in solving matrix eqn (76) is characterized by $N_1 = h^3$ and $N_2 = h^2$.

If the active load is associated with a two-dimensional representation τ_r, say τ_9, we have four corresponding columns in matrix B_*, namely, b_{*9}, b_{*10}, b_{*11} and b_{*12}. However, it is necessary to solve only one subsystem of order $2m$

$$\Lambda_9 \begin{bmatrix} y_9 & 0 \\ 0 & y_{10} \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \tag{111}$$

because $y_{11} = y_9$ and $y_{12} = y_{10}$. Finally,

$$x_{*s} = u_s \otimes y_s, \quad 9 \leqslant s \leqslant 12. \tag{112}$$

6.4. Blocks Λ_r, which relate to the one-dimensional irreducible representations, have a simple mechanical interpretation. In the case under consideration $\Lambda_1, \ldots, \Lambda_8$ can be thought of as matrices (stiffness matrices, for instance) of the fundamental primitive S_1 of system D_{4h} with eight different boundary conditions along the cut edges lying in symmetry planes σ_h, $\sigma_v^{(1)}$ and $\sigma_v^{(2)}$. These conditions are known as symmetric and antisymmetric (skewsymmetric). To define them let us assume that a certain symmetry plane P is spanned by the coordinate axes y and z and that node A lies in P. Then the restrictions imposed on motion of node A

$$u_x^I = \phi_y^I = \phi_z^I = 0 \tag{113}$$

are called symmetric while

$$u_y^I = u_z^I = \phi_x^I = 0 \tag{114}$$

are called antisymmetric. (Here u_i^I and ϕ_i^I are respectively translational and rotational degrees of freedom, $i = x, y, z$.) According to this definition Λ_1 and Λ_3 are associated with the fundamental primitive S_1 with respectively symmetric and antisymmetric boundary conditions along all of its cut edges. All blocks $\Lambda_1, \ldots, \Lambda_8$ are interpreted in Table 11. The active loads, applied to the fundamental primitive and described by subvectors $c_r \equiv c_{r1}$ (78), can be treated analogously.

Hermitian matrices Λ_9 and Λ_{10} are associated with two primitives joined together and having special complex (i.e. non-real) boundary conditions along the cut edges. Therefore, in general, the response of system D_{4h} cannot be expressed as a superposition of partial responses of its uncoupled primitives.

Finally, Theorem 4 permits the following mechanical interpretation: the original FSS composed of h primitives and containing mh degrees of freedom is replaced by H uncoupled subsystems $S^{(r)}$ corresponding to H irreducible representations τ_r, $r = 1, \ldots, H$, of the associated symmetry group G. Subsystem $S^{(r)}$ has mn_r degrees of freedom and is composed of n_r primitives of the FSS ($1 \leqslant n_r \leqslant 5$). Its boundary conditions along the cut edges are determined in accordance with τ_r and may be described by real-valued as well as by complex-valued functions. Subsystem $S^{(r)}$ is subjected to n_r special loads whose load vectors $c_{r\gamma}$, $\gamma = 1, \ldots, n_r$ are obtained by projecting the original load vector b_* onto the vector space $V^{(r)}$ spanned by mn_r column-vectors $u_{r\gamma\alpha} \otimes I_m$ ($\alpha = 1, \ldots, n_r$) of matrix $U_* = U \otimes I_m$. The response of the original FSS is found as a superposition of partial responses of all subsystems $S^{(r)}$, $r = 1, \ldots, H$. If the active load applied to the FSS follows a particular irreducible representation τ_s then the load vector b_* lies in the vector space $V^{(s)}$. Hence all other subsystems $S^{(r)}$, $r \neq s$, are unloaded and do not participate in the total FSS response. This is a typical case in the analysis of the FSS and usually τ_s is one-dimensional but not necessarily the unity representation τ_u.

6.5. It is easy now to trace an analogy between the symmetry approach and the modal analysis. For the sake of simplicity we consider the latter in the static interpretation: Let

Table 11. Symmetric and antisymmetric boundary conditions at the cut edges of the D_{4h} fundamental primitive

#	MATRICES Λ_i	SYMMETRY PLANES		
		HORIZONTAL σ_h	VERTICAL $\sigma_v^{(2)}$	VERTICAL $\sigma_v^{(1)}$
1	$\Lambda_1 = A_1 + A_5 + A_9 + A_{12} + A_{13} + A_{16}$	S = SYMMETRIC	S	S
2	$\Lambda_2 = A_1 + A_5 - A_9 - A_{12} - A_{13} - A_{16}$	S	A = ANTISYMMETRIC	A
3	$\Lambda_3 = A_1 - A_5 + A_9 + A_{12} - A_{13} - A_{16}$	A	A	A
4	$\Lambda_4 = A_1 - A_5 - A_9 - A_{12} + A_{13} + A_{16}$	A	S	S
5	$\Lambda_5 = A_1 + A_5 + A_9 - A_{12} + A_{13} - A_{16}$	S	S	A
6	$\Lambda_6 = A_1 + A_5 - A_9 + A_{12} - A_{13} + A_{16}$	S	A	S
7	$\Lambda_7 = A_1 - A_5 - A_9 + A_{12} - A_{13} + A_{16}$	A	A	S
8	$\Lambda_8 = A_1 - A_5 + A_9 - A_{12} + A_{13} - A_{16}$	A	S	A

the matrix equation

$$Ax = b \tag{115}$$

correspond to a certain linear mechanical system with n degrees of freedom. Matrix A is real and symmetric, it has full set of eigenpairs (λ_i, u_i), $i = 1, \ldots, n$. Denote by Λ the spectral matrix of A, that is, the diagonal matrix of eigenvalues λ_i and by U the fundamental matrix of A, i.e. the matrix whose columns are the eigenvectors u_i normalized to the unit length. Matrix U is orthonormal: $U^{-1} = U^{\mathrm{T}}$. Let us present matrix A as the following triple product

$$A = U\Lambda U^{\mathrm{T}}, \tag{116}$$

substitute it into (115) and premultiply the result by U^{T}:

$$\Lambda U^{\mathrm{T}}x = U^{\mathrm{T}}b$$

or

$$\Lambda y = c, \tag{117}$$

where

$$y = U^{\mathrm{T}}x \quad \text{and} \quad c = U^{\mathrm{T}}b. \tag{118}$$

Having obtained the solution of the diagonal system (117) one can find

$$x = Uy = \sum_{i=1}^{n} y_i u_i, \tag{119}$$

which means that the modal superposition is a weighted superposition of the eigenvectors u_i and the weights y_i are solutions of (117). Clearly, components c_i of the vector c (118) are dot products of the eigenvectors and the load vector b:

$$c_i = u_i^{\mathrm{T}}b \equiv u_i \cdot b, \quad i = 1, \ldots, n. \tag{120}$$

Hence, if b is parallel to a certain eigenvector u_j (we can say that the active load follows u_j), then it is perpendicular to all others and $c_i = 0$ for all $i \neq j$. Usually the modal superposition is limited to, say, p low modes under the assumption that the load vector b belongs to a vector subspace spanned by eigenvectors $u_1, u_2, \ldots, u_p$. Therefore $c_i = 0$ for $i = p+1, p+2, \ldots, n$.

The great advantage of the modal analysis lies in the fact that it is applicable to any linear system. Its disadvantage is revealed in the requirement to solve a full or a partial eigenvalue problem. The symmetry approach has the exact opposite advantage and disadvantage: the method is explicit, no eigenvalue problem has to be solved; however, it is applicable to symmetric systems only. It is interesting to note that if the primitives of the FSS have one degree of freedom each ($m = 1$) and the corresponding symmetry group G possesses one-dimensional irreducible representations only, then both methods are identical. In this case matrix Λ_* becomes diagonal and consists of h eigenvalues of $A_* = \Sigma_{i=1}^{h} Q(g_i)a_i$, while matrix $U_* = U \otimes I_1 \equiv U$ is the fundamental matrix, hence the full eigenvalue problem has an explicit solution. Several matrices possessing this property are given in Dinkevich (1986).

7. SYMMETRY APPROACH IN STRUCTURAL ANALYSIS OF A VACUUM VESSEL

7.1. To illustrate the symmetry technique we will trace here simplifications in the structural analysis of the vacuum vessel of a fusion machine subjected to a set of loads with different types of symmetry. The vacuum vessel (chamber camera) is a toroidal shell with a circular (or D-shape) cross-section. It has many horizontal and vertical penetration ports which are located cyclically symmetric in the circumferential direction. These ports together with the supports (usually placed between the horizontal ports) reduce geometrical symmetry of the vessel from $D_{\infty h}$ (axisymmetry) to D_{nh}, where $n \sim 10$–20. To simplify the problem we assume $n = 4$. Hence, the structural model of the vacuum vessel has symmetry D_{4h} and consists of $h = 4n = 16$ primitives which are 45° sectors with a half circular cross-section; see Figs 2 and 3 and compare with Fig. 1.

We begin with the case when there is no parametric load and the active load is arbitrary. In this case matrix equation $A_* x_* = b_*$ with matrix A_* given by (94) is explicitly decomposed to the following form

$$\begin{bmatrix} \Lambda_1 &&&&&&&&&&& \\ & \Lambda_2 &&&&&&&&&& \\ && \Lambda_3 &&&&&&&&& \\ &&& \Lambda_4 &&&&&&&& \\ &&&& \Lambda_5 &&&&&&& \\ &&&&& \Lambda_6 &&&&&& \\ &&&&&& \Lambda_7 &&&&& \\ &&&&&&& \Lambda_8 &&&& \\ &&&&&&&& \Lambda_9 &&& \\ &&&&&&&&& \Lambda_9 && \\ &&&&&&&&&& \Lambda_{10} & \\ &&&&&&&&&&& \Lambda_{10} \end{bmatrix} \begin{bmatrix} y_1 \\ \hline y_2 \\ \hline y_3 \\ \hline y_4 \\ \hline y_5 \\ \hline y_6 \\ \hline y_7 \\ \hline y_8 \\ \hline y_{9,1} \\ \hline y_{9,2} \\ \hline y_{10,1} \\ \hline y_{10,2} \end{bmatrix} = \begin{bmatrix} c_1 \\ \hline c_2 \\ \hline c_3 \\ \hline c_4 \\ \hline c_5 \\ \hline c_6 \\ \hline c_7 \\ \hline c_8 \\ \hline c_{9,1} \\ \hline c_{9,2} \\ \hline c_{10,1} \\ \hline c_{10,2} \end{bmatrix} \tag{121}$$

where $\Lambda_1, \ldots, \Lambda_8$ are determined by (103)–(104) and Λ_9 and Λ_{10} by (105)–(106). Subvectors

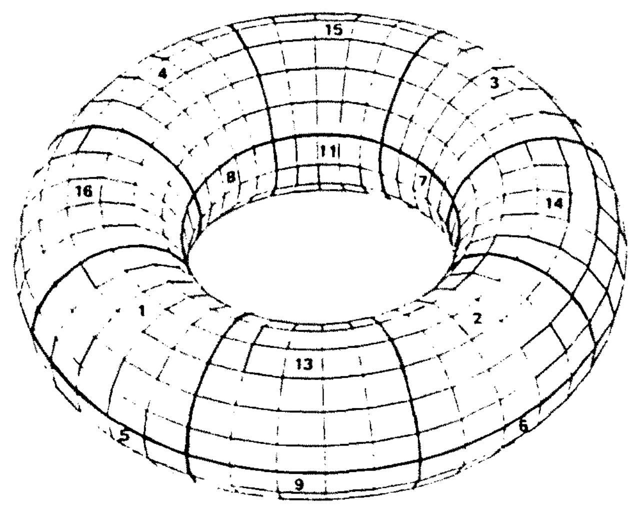


Fig. 2. Toroidal vacuum vessel (penetration ports and supports are not shown). System D_{4h}.

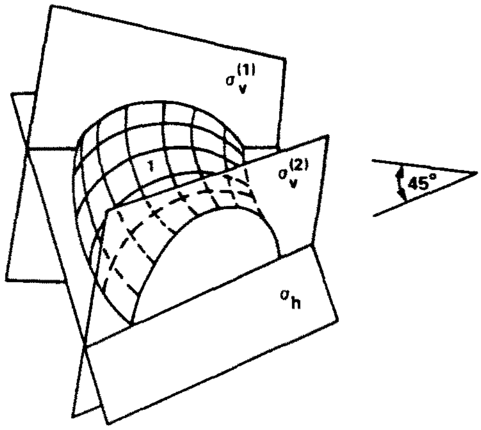


Fig. 3. Fundamental primitive of system D_{4h}.

$c_1, \ldots, c_8$ (of order m) are calculated by

$$\begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_8 \end{bmatrix} = \frac{1}{4} \sum_{\mu=1}^{n=4} (S_\mu \otimes I_m) \begin{bmatrix} b_\mu \\ b_{\mu+4} \\ b_{\mu+8} \\ b_{\mu+12} \end{bmatrix}. \tag{122}$$

Subvectors $c_{9,\gamma}$ and $c_{10,\gamma}$, $\gamma = 1, 2$, are of order $2m$. They may be presented in the following form [compare with (107)]:

$$c_{9,1} = \begin{bmatrix} c_{9,11} \\ c_{9,12} \end{bmatrix}, \quad c_{9,2} = \begin{bmatrix} \bar{c}_{9,12} \\ \bar{c}_{9,11} \end{bmatrix}$$

$$c_{10,1} = \begin{bmatrix} c_{10,11} \\ c_{10,12} \end{bmatrix}, \quad c_{10,2} = \begin{bmatrix} \bar{c}_{10,12} \\ \bar{c}_{10,11} \end{bmatrix}. \tag{123}$$

Hence, it is sufficient to compute only $c_{9,1}$ and $c_{10,1}$:

$$\begin{bmatrix} c_{9,1} \\ \hline c_{10,1} \end{bmatrix} \equiv \begin{bmatrix} c_{9,11} \\ \hline c_{9,12} \\ \hline c_{10,11} \\ \hline c_{10,12} \end{bmatrix} = \frac{1}{2\sqrt{2}} \sum_{\mu=1}^{n=4} \bar{\varepsilon}_{\mu-1} \left(\begin{bmatrix} 1 & 1 & & \\ & & 1 & 1 \\ 1 & -1 & & \\ & & -1 & 1 \end{bmatrix} \otimes I_m \right) \begin{bmatrix} b_\mu \\ \hline b_{\mu+4} \\ \hline b_{\mu+8} \\ \hline b_{\mu+12} \end{bmatrix}. \tag{124}$$

Finally one finds vector x_*:

$$\begin{bmatrix} x_\mu \\ \hline x_{\mu+4} \\ \hline x_{\mu+8} \\ \hline x_{\mu+12} \end{bmatrix} = \frac{1}{4} \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \otimes I_m \right) \left(\begin{bmatrix} y_1 \\ \hline y_2 \\ \hline y_3 \\ \hline y_4 \end{bmatrix} + (-1)^{\mu-1} \begin{bmatrix} y_5 \\ \hline y_6 \\ \hline y_7 \\ \hline y_8 \end{bmatrix} \right)$$

$$+ \frac{1}{\sqrt{2}} \mathrm{Re} \left(\varepsilon_{\mu-1} \begin{bmatrix} y_{9,11} + y_{10,11} \\ \hline y_{9,11} - y_{10,11} \\ \hline y_{9,12} - y_{10,12} \\ \hline y_{9,12} + y_{10,12} \end{bmatrix} \right), \quad \mu = 1,2,3,4. \tag{125}$$

7.2. The vacuum vessel is designed to confine the plasma ignited in it, therefore the majority of its loadings is associated with plasmas, that is, with several scenarios of plasma motions and disruptions. The plasma induces the parametric and the active loads on the vacuum vessel. To simplify these plasma-induced loadings we will associate the temperature distributions throughout the vessel with the parametric loads only. Then the active loads will be the Lorentz's electromagnetic forces, that is, the pressure gradients, ∇p, which are equal to the cross products of the eddy current, J, induced in the walls, and the magnetic field, B, created to confine the plasma inside the vessel: $\nabla p = J \times B$.

One set of such loads corresponds to the horizontal (inboard) axisymmetic plasma motion. These loads, both parameteric and active, have symmetry $D_{\infty h}$ whose intersection with geometric symmetry D_{4h} gives the actual symmetry of the model D_{4h}. The active load follows the unity irreducible representation of group D_{4h}. This case was discussed in the previous section.

7.3. Another set of plasma loads relates to the axisymmetric vertical plasma motion. Clearly, in this case the temperature distributions above and below the midplane are not identical, therefore the parametric load does not have a horizontal plane of symmetry. Its symmetry is $C_{\infty v}$, hence, the actual symmetry of the model is $D_{4h} \cap C_{\infty v} = C_{4v}$. In such a case the vacuum vessel is divided into $h = 2n = 8$ primitives which are 45° sectors with a circular cross-section, that is, they are double the size of those in Figs 2 and 3.

The stiffness matrix A_* takes the form

$$A_*(C_{4v}) = \left[\begin{array}{cccc|cccc} A_1 & & & & A_5 & & & A_8 \\ & A_1 & & & A_8 & A_5 & & \\ & & A_1 & & & A_8 & A_5 & \\ & & & A_1 & & & A_8 & A_5 \\ \hline & & & & A_1 & & & \\ & & & & & A_1 & & \\ & & & & & & A_1 & \\ & & SYMMETRIC & & & & & A_1 \end{array}\right]. \tag{126}$$

Its blocks are of order $2m$. They are symmetric according to Table 6. Group C_{4v} has four one-dimensional irreducible representations and a single two-dimensional one, therefore matrix A_* (126) may be transformed by the similarity transformation to the following block diagonal form

$$\Lambda_* = \begin{bmatrix} \Lambda_1 & & & & & \\ & \Lambda_2 & & & & \\ & & \Lambda_3 & & & \\ & & & \Lambda_4 & & \\ & & & & \Lambda_5 & \\ & & & & & \Lambda_5 \end{bmatrix}. \tag{127}$$

The order of $\Lambda_1, \ldots, \Lambda_4$ is $2m$ while $\dim \Lambda_5 = 4m$. The active load follows the unity representation which is τ_1 for group C_{nv}, hence, in accordance with (100), $c_2 = \cdots = c_5 = 0$. Thus one has to solve only one $2m \times 2m$ subsystem $\Lambda_1 y_1 = c_1$, where

$$\Lambda_1 = A_1 + A_5 + A_8, \quad c_1 = b. \tag{128}$$

The total system response is determined by

$$x_1 = \cdots = x_8 = (1/2\sqrt{2})y_1. \tag{129}$$

7.4. As an alternative to the above case let us consider the vacuum vessel model C_{4h} in which the system is divided into another set of $h = 8$ primitives. They are 90° sectors with a half circular cross-section and each has $2m$ degrees of freedom, the same as primitives of C_{4v}. The stiffness matrix A_* has the following structure

$$A_*(C_{4h}) = \left[\begin{array}{cccc|cccc} A_1 & A_2 & & A_2^T & A_5 & A_6 & & A_6^T \\ & A_1 & A_2 & & A_6^T & A_5 & A_6 & \\ & & A_1 & A_2 & & A_6^T & A_5 & A_6 \\ & & & A_1 & A_6 & & A_6^T & A_5 \\ \hline & & & & A_1 & A_2 & & A_2^T \\ & & & & & A_1 & A_2 & \\ & & & & & & A_1 & A_2 \\ \multicolumn{4}{l}{SYMMETRIC} & & & & A_1 \end{array}\right]. \tag{130}$$

All its blocks are of order $2m$, blocks A_1 and A_5 are symmetric. Group C_{4h} (see Table 5) has eight one-dimensional irreducible representations τ_1–τ_8. Based on them matrix A_*, (130) is transformed to

$$\Lambda_* = \begin{bmatrix} \Lambda_1 & & & & & & & \\ & \Lambda_2 & & & & & & \\ & & \Lambda_3 & & & & & \\ & & & \Lambda_4 & & & & \\ & & & & \Lambda_5 & & & \\ & & & & & \Lambda_6 & & \\ & & & & & & \Lambda_7 & \\ & & & & & & & \Lambda_8 \end{bmatrix} \tag{131}$$

and $\dim \Lambda_r = 2m$, $r = 1, \ldots, 8$. Blocks Λ_2, Λ_4, Λ_6 and Λ_8 are real and symmetric because they correspond to real irreducible representations. Others are Hermitian and form complex conjugate pairs: Λ_1 and $\Lambda_3 = \bar{\Lambda}_1$, Λ_5 and $\Lambda_7 = \bar{\Lambda}_5$. The active load follows the unity representation which in this case is τ_4, hence, it is necessary to solve subsystem $\Lambda_4 y_4 = c_4$, where

$$\Lambda_4 = A_1 + A_2 + A_2^T + A_5 + A_6 + A_6^T, \quad c_4 = b. \tag{132}$$

Having obtained its solution we find

$$x_1 = \cdots = x_8 = (1/2\sqrt{2})y_4. \tag{133}$$

7.5. In the above Subsections 7.2–7.4 we have suggested that the active loads followed the unity representation of groups D_{4h}, C_{4v} or C_{4h} indicating that these loads had symmetry D_{4h}, C_{4v} or C_{4h}, respectively. In fact, those of the electromagnetic forces which are induced on the walls of the penetration ports do not possess horizontal and vertical planes of reflection. Instead they have a center of inversion and create torsion moments around the ports. Hence, strictly speaking, symmetry of the active load is only C_4. Then in accordance with Figs 1 and 2 primitives S_1–S_4 are subjected to the same load, say, h_1, primitives S_5–S_8 to another load h_5, while primitives S_9–S_{12} and S_{13}–S_{16} are subjected to h_9 and h_{13}, respectively, where h_1, h_5, h_9 and h_{13} are arbitrary load vectors of order m. The total load vector is

$$h_*^T = 1/4[h_1h_1h_1h_1\ h_5h_5h_5h_5\ h_9h_9h_9h_9\ h_{13}h_{13}h_{13}h_{13}]_{16m}. \tag{134}$$

A factor 1/4 is introduced to satisfy eqn (100) in a case where $h_1 = h_5 = h_9 = h_{13}$, (see the footnote on p. 1239). Substituting (134) into (78) and invoking Table 9 we find that there are four non-zero subvectors c_r in the right side of eqn (121):

$$\begin{aligned} c_1 &= 1/4(h_1+h_5+h_9+h_{13}) \\ c_2 &= 1/4(h_1+h_5-h_9-h_{13}) \\ c_5 &= 1/4(h_1-h_5-h_9+h_{13}) \\ c_6 &= 1/4(h_1-h_5+h_9-h_{13}). \end{aligned} \tag{135}$$

Having solved four $m \times m$ subsystems

$$\Lambda_1 y_1 = c_1, \quad \Lambda_2 y_2 = c_2, \quad \Lambda_5 y_5 = c_5 \quad \text{and} \quad \Lambda_6 y_6 = c_6, \tag{136}$$

where $\Lambda_1, \ldots, \Lambda_6$ are given in Table 11, and introducing their solutions into $x_* = U_* y_*$, we obtain

$$\begin{aligned} x_1 = x_3 = x_5 = x_7 &= 1/4[(y_1+y_2)+(y_5+y_6)] \\ x_2 = x_4 = x_6 = x_8 &= 1/4[(y_1+y_2)-(y_5+y_6)] \\ x_9 = x_{11} = x_{13} = x_{15} &= 1/4[(y_1-y_2)+(y_5-y_6)] \\ x_{10} = x_{12} = x_{14} = x_{16} &= 1/4[(y_1-y_2)-(y_5-y_6)]. \end{aligned} \tag{137}$$

7.6. Now we assume that symmetry of the parametric loads is also C_4. Then the actual symmetry of the vessel model is $D_{4h} \cap C_4 = C_4$. It consists of $h = n = 4$ primitives which are 90° sectors with a circular cross-section. Each primitive has $4m$ degrees of freedom. Now the stiffness matrix

$$A_*(C_4) = \begin{bmatrix} A_1 & A_2 & & A_2^T \\ & A_1 & A_2 & \\ & & A_2 & A_2 \\ \text{SYMMETRIC} & & & A_1 \end{bmatrix} \tag{138}$$

can be explicitly block diagonalized to

$$\Lambda_* = \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \Lambda_3 & \\ & & & \Lambda_4 \end{bmatrix}, \tag{139}$$

where $\dim \Lambda_1 = \cdots = \dim \Lambda_4 = 4m$. Since the active load follows the unity representation of C_4 which is τ_4, one has to solve the subsystem $\Lambda_4 y_4 = c_4$, where

$$\Lambda_4 = A_1 + A_2 + A_2^T, \quad c_4 = h. \tag{140}$$

The system response is determined by

$$x_1 = \cdots = x_4 = (1/2)y_4. \tag{141}$$

7.7. Finally, we consider a non-axisymmetric plasma motion and disruption, and assume that symmetry of plasma loads is equal to C_{1v}, i.e. it consists of one vertical plane of reflection. If this plane does not coincide with one of the vertical symmetry planes of the vacuum vessel, then the model has no symmetry: $D_{4h} \cap C_{1v} = C_1$. For such a case it is necessary to model the whole vessel. If $D_{4h} \cap C_{1v} = C_{1v}$ the model is divided into two primitives which are 180°-sectors with $8m$ degrees of freedom each. Then

$$A_*(C_{1v}) = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_2 & A_1 \end{array}\right], \quad A_1^T = A_1,\ A_2^T = A_2 \tag{142}$$

and it is necessary to solve a subsystem

$$(A_1 + A_2)y_1 = b \tag{143}$$

of order $8m$. The total response of system C_{1v} is described by

$$x_1 = x_2 = (1/\sqrt{2})b. \tag{144}$$

7.8. The Compact Ignition Tokamak (CIT) vacuum vessel structure with 18 radial penetration ports and 20 supports is the FSS D_{2h}. It possesses two vertical and one horizontal symmetry planes and one vertical and two horizontal axes c_2 lying at the intersections of the planes, see Fig. 4. In order to increase the degree of symmetry of the analytical model,

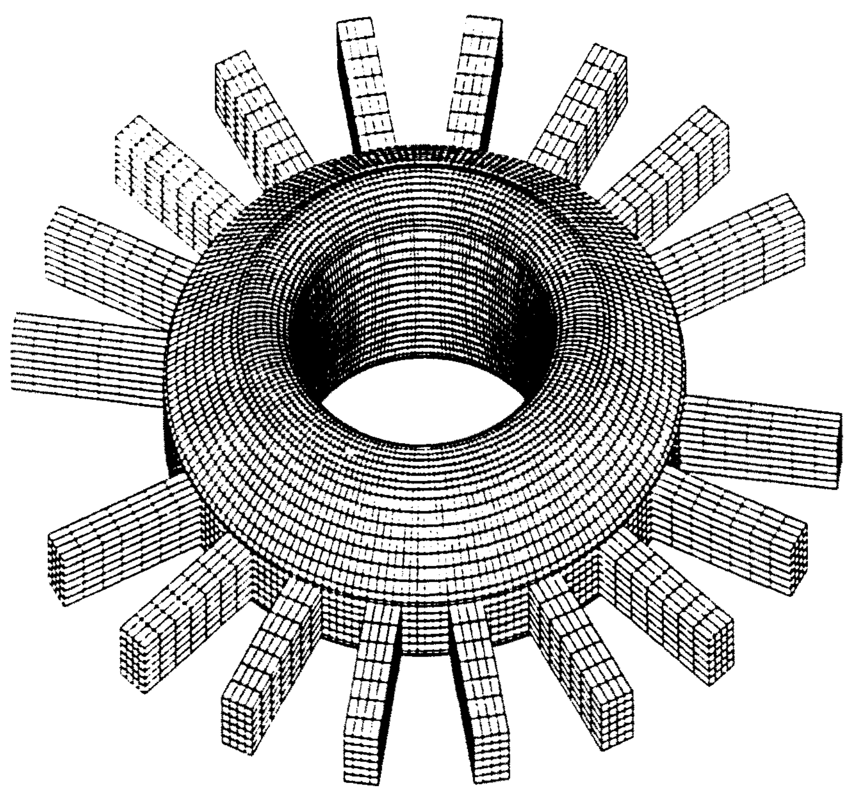


Fig. 4. The CIT vacuum vessel model (vertical ports and supports are not shown). System D_{2h}.

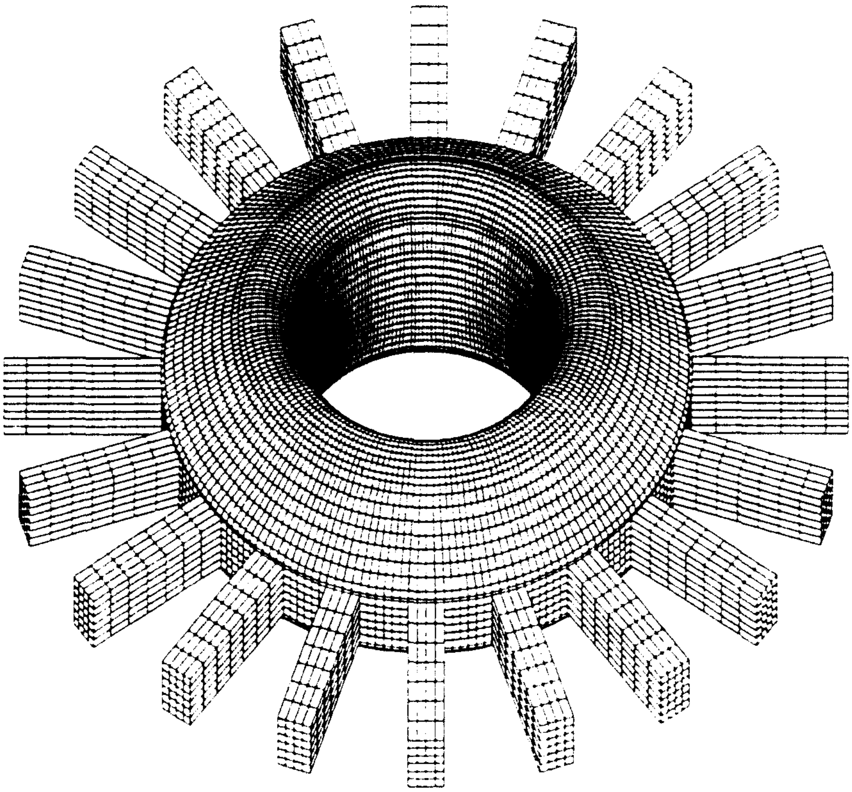


Fig. 5. Analytical model of the CIT vacuum vessel. System D_{20h}.

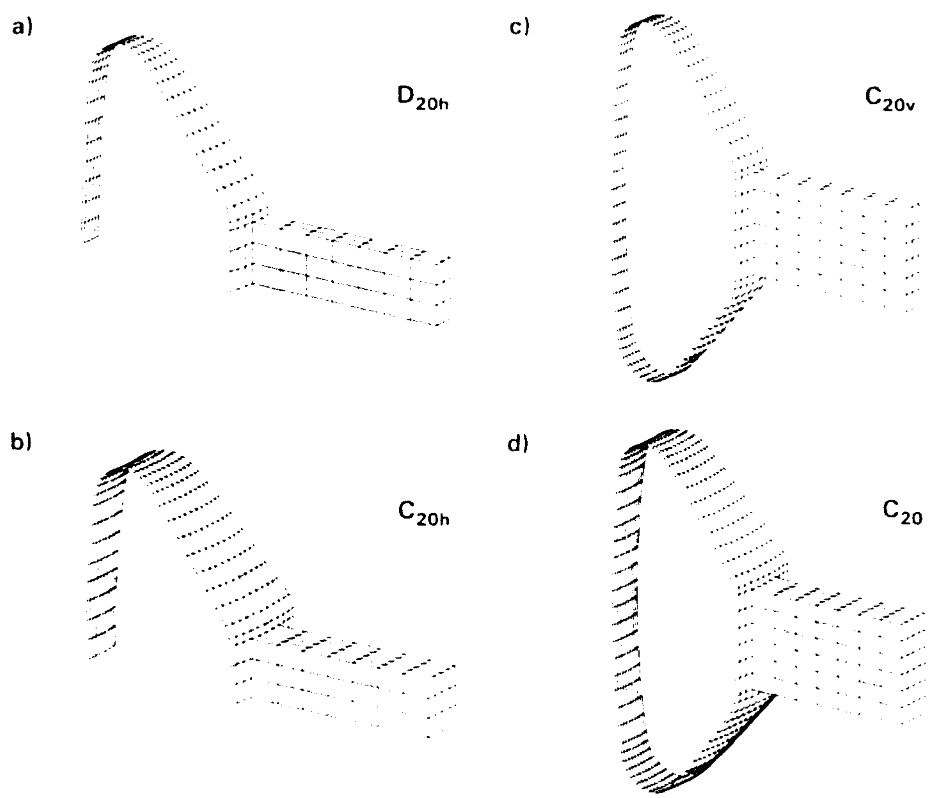


Fig. 6. Primitives of the CIT vacuum vessel models: (a) primitives of model D_{20h}, (b) primitives of model C_{20h}, (c) primitives of model C_{20v}, and (d) primitives of model C_{20}.

two artificial radial ports were added†. By adding these ports the vacuum vessel structure can be represented by the FSS D_{20h}; see Fig. 5. However, with regard to the applied plasma disruption loads the actual symmetry of the model was reduced to C_{20v} and then to C_{20} and both models were used in the structural analysis of the CIT vacuum vessel by utilizing the MSC/NASTRAN finite element computer program. The fundamental primitives of the vacuum vessel models D_{20h}, C_{20h}, C_{20v} and C_{20} are shown in Fig. 6.

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† This led to some changes in structural responses; however, all changes were localized to the areas around the added ports.